J. Mech. Phys. Solids, 1967, Vol. 15, pp. 299 to 309. Pergamon Press Ltd. Printed in Great Britain.

A GENERALIZED DYNAMICAL THEORY OF THERMOELASTICITY

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(*Received* 15*th March* 1967)

Summary

In this work a generalized dynamical theory of thermoelasticity is formulated using a form of the heat transport equation which includes the time needed for acceleration of the heat flow. The theory takes into account the coupling effect between temperature and strain rate, but the resulting coupled equations are both hyperbolic. Thus, the paradox of an infinite velocity of propagation, inherent in the existing coupled theory of thermoelasticity, is eliminated. A solution is obtained using the generalized theory which compares favourably with a known solution obtained using the conventional coupled theory.

1. Introduction

The governing equations for the displacement and temperature fields, as given by the linear dynamical theory of thermoelasticity, consist of the following two coupled partial differential equations

$$\text{Equation of motion } \rho\, \ddot{u}_i = (\lambda + \mu)\, u_{j,\,ij} + \mu u_{i,\,jj} - (3\lambda + 2\mu)\, \alpha\, T_{,i} \tag{1}$$

$$\text{Energy equation} \quad k\, T_{,ii} = \rho c_E\, \dot{T} + (3\lambda + 2\mu)\, \alpha\, T_0\, \dot{\epsilon}_{kk}. \tag{2}$$

The displacement field is governed by a wave-type equation, (1) and the temperature field is governed by a diffusion type equation, (2). The properties of the latter are such that a portion of the solution extends to infinity. That is, if an isotropic, homogeneous elastic continuum is subjected to a mechanical or thermal disturbance, the effect of the disturbance will be felt instantaneously at distances infinitely far from its source. Moreover, this effect will be felt in both the temperature and the displacement fields, since the governing equations are coupled.

Physically, this means that a portion of the disturbance has an infinite velocity of propagation. Such behaviour is physically inadmissible and contradicts existing theories of heat transport mechanisms. This apparent paradox in the existing theory of thermoelasticity has been discussed by other authors, such as Boley (1964) for example. The theory presented here eliminates the paradox of an infinite velocity of propagation by employing a more general functional relation between heat flow and temperature gradient than is used in the existing theory.

2. Formulation of the Theory

The energy equation for a thermally conducting elastic solid subjected to small rains and small temperature changes is given by

$$\sigma_{ij}\,\dot{\epsilon}_{ij} + \rho\,T\,\dot{s} = \rho\,\dot{e} \tag{3}$$

where

$$\rho\,T\,\dot{s} = -\,q_{i,\,i} \tag{4}$$

and σ_{ij} is the stress tensor, ϵ_{ij} is the small strain tensor, ρ is the mass density, e is the internal energy density, T is the absolute temperature, s is the entropy density, and q_i is the heat flux vector.

As shown in BOLEY and WEINER (1960), the principle of positive entropy-density production in an elastic continuous solid implies that q_i and the temperature gradient in the solid $T,_i$ cannot take on arbitrary values, but that some functional relationship must exist between them. In the usual derivation of the coupled theory of thermoelasticity this functional relationship is taken to be linear, having the general form

$$q_i = b\,T,_i + B_{ij}\,T,_j. \tag{5}$$

For an isotropic elastic solid this reduces to the wellknown Fourier law of heat conduction

$$q_i = -\,k\,T,_i \tag{6}$$

where the scalar k is the thermal conductivity of the solid. The development of the theory presented herein deviates from previous work at this point by assuming a more general functional relationship between q_i and $T,_i$.

The premise of local reversibility is inherent in the derivation of the theory leading to the energy equation, as stated above. This same premise is used in the work by ONSAGER (1931) on reciprocal relations in irreversible processes. Onsager points out that the form of the heat conduction equation given by (6) results in an apparent contradiction to this premise, but adds that the objection is removed when we recognize that (6) '. . . . is only an approximate description of the process of conduction, neglecting the time needed for acceleration of the heat flow '.

The most general, tensorially valid, linear relation between q_i and $T,_i$ which takes into account the acceleration of the heat flow is of the form

$$q_i + a\dot{q}_i + A_{ij}\,\dot{q}_j = b\,T,_i + B_{ij}\,T,_j \tag{7}$$

where a, A_{ij}, b and B_{ij} are material properties of the medium. For an isotropic elastic solid this reduces to

$$q_i + \tau_0\,\dot{q}_i = -\,k\,T,_i \tag{8}$$

where τ_0, *the relaxation time*, represents the time-lag needed to establish steady-state heat conduction in an element of volume when a temperature gradient is suddenly imposed on that element. Equation (8) represents the modified form of the heat conduction equation which has been given a clear physical interpretation by CHESTER (1963).

It is convenient at this point to introduce the Helmholtz free-energy function $\phi\,(\epsilon_{ij}, T)$ defined as

$$\phi\,(\epsilon_{ij}, T) = e\,(\epsilon_{ij}, T) - T\,s\,(\epsilon_{ij}, T). \tag{9}$$

It follows from (3) and (9) and the relation

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial\epsilon_{ij}}\,\dot{\epsilon}_{ij} + \frac{\partial\phi}{\partial T}\,\dot{T} \tag{10}$$

that the stress tensor σ_{ij} and the entropy s can be expressed in terms of the free-energy function as

$$\sigma_{ij} = \rho \frac{\partial \phi}{\partial \epsilon_{ij}}, \tag{11}$$

$$s = - \frac{\partial \phi}{\partial T}. \tag{12}$$

Substituting (12) into (4) yields

$$q_{i,i} = \rho T \left(\frac{\partial^2 \phi}{\partial T^2} \dot{T} + \frac{\partial^2 \phi}{\partial \epsilon_{ij} \partial T} \dot{\epsilon}_{ij} \right). \tag{13}$$

Combining (8) and (13), the following form of the energy equation in terms of the free-energy function is obtained

$$kT,_{ii} = - \rho T \left[\frac{\partial^2 \phi}{\partial T^2} (\dot{T} + \tau_0 \ddot{T}) + \frac{\partial^2 \phi}{\partial \epsilon_{ij} \partial T} (\dot{\epsilon}_{ij} + \tau_0 \ddot{\epsilon}_{ij}) \right]$$

$$- \rho \tau_0 \left[(\dot{T})^2 \left(\frac{\partial^2 \phi}{\partial T^2} + T \frac{\partial^3 \phi}{\partial T^3} \right) + (\dot{\epsilon}_{ij} \dot{T}) \left(\frac{\partial^2 \phi}{\partial \epsilon_{ij} \partial T} + 2T \frac{\partial^3 \phi}{\partial \epsilon_{ij} \partial T^2} \right) + (\dot{\epsilon}_{ij})^2 \left(T \frac{\partial^3 \phi}{\partial^2 \epsilon_{ij} \partial T} \right) \right]. \tag{14}$$

The last bracketed set of terms on the right-hand side can be neglected within the framework of the usual assumptions of the linear theory. Thus, the energy equation becomes

$$kT,_{ii} = - \rho T \left[\frac{\partial^2 \phi}{\partial T^2} (\dot{T} + \tau_0 \ddot{T}) + \frac{\partial^2 \phi}{\partial \epsilon_{ij} \partial T} (\dot{\epsilon}_{ij} + \tau_0 \ddot{\epsilon}_{ij}) \right]. \tag{15}$$

In a manner similar to that of Boley and Weiner, the free energy function can be expanded in power series of the three strain invariants and the dimensionless temperature change θ, where

$$\theta = (T - T_0)/T_0 \tag{16}$$

and where T_0 is the stress-free temperature of the body. With this expansion and with the usual definition of the specific heat at constant deformation, given as

$$c_E = - T \frac{\partial^2 \phi}{\partial T^2}, \tag{17}$$

the stress–strain–temperature relation (11) becomes

$$\sigma_{ij} = a_1 \delta_{ij} + a_2 (2 I_\varepsilon \delta_{ij}) + a_3 (I_\varepsilon \delta_{ij} - \epsilon_{ij}) + a_7 (\theta \delta_{ij}) \tag{18}$$
$$+ \text{ higher order terms.}$$

The energy equation (15) becomes

$$kT,_{ii} = - \frac{T}{T_0^2} \{ 2a_{13} + 2a_{17} I_\varepsilon + 6a_{25} \theta + \ldots \} (\dot{T} + \tau_0 \ddot{T}) - \frac{T}{T_0} \{ a_7 \delta_{ij}$$

$$+ a_{11} (\delta_{ij} I_\varepsilon - \epsilon_{ij}) + 2a_{14} I_\varepsilon \delta_{ij} + 2a_{17} \theta \delta_{ij} + a_{19} [I_\varepsilon (\delta_{ij} I_\varepsilon - \epsilon_{ij}) + II_\varepsilon \delta_{ij}]$$

$$+ a_{21} (\epsilon_{ij} \epsilon_{jk} - \epsilon_{ij} I_\varepsilon + \delta_{ij} II_\varepsilon) + 3a_{24} I_\varepsilon^2 \delta_{ij} + \ldots \} (\dot{\epsilon}_{ij} + \tau_0 \ddot{\epsilon}_{ij}) \tag{19}$$

and the specific heat c_E is given as

$$c_E = \frac{-T}{\rho T_0^2} (2a_{13} + 2a_{17} I_\varepsilon + 6a_{25} \theta + \ldots) \tag{20}$$

where the various a_m's are coefficients in the series for ϕ, and where I_ε and II_ε are the first two pr'ncipal strain invariants.

Linearized form of the governing equations

The usual linear theory of thermoelasticity is obtained by considering the case where only terms linear in strain ϵ_{ij} and temperature change θ are retained in (18) and (19). This allows the coefficients a_1, a_2, a_3 and a_7 to be determined in terms of the Lamé elastic constants, λ and μ, with the results

$$\left.\begin{aligned} a_1 &= 0, \qquad & a_2 &= (\lambda + 2\mu)/2, \\ a_3 &= -\,2\mu, \qquad & a_7 &= -\,(3\lambda + 2\mu)\,\alpha T_0. \end{aligned}\right\} \tag{21}$$

Substitution of these results into (18) yields the familar linear constitutive equation

$$\sigma_{ij} = \lambda\,\epsilon_{kk}\,\delta_{ij} + 2\mu\,\epsilon_{ij} - (3\lambda + 2\mu)\,\alpha\,(T - T_0)\,\delta_{ij}. \tag{22}$$

Assuming a constant specific heat and small θ, the linearized energy equation becomes

$$kT,_{ii} = \rho c_E\,(\dot{T} + \tau_0\,\ddot{T}) + (3\lambda + 2\mu)\,\alpha\,T_0\,(\dot{\epsilon}_{kk} + \tau_0\,\ddot{\epsilon}_{kk}). \tag{23}$$

The equation of motion for a linear elastic continuum, in the absence of body forces, is that given by (1) and repeated here

$$\rho\,\ddot{u}_i = (\lambda + \mu)\,u_{j,\,ij} + \mu\,u_{i,jj} - (3\lambda + 2\mu)\,\alpha\,T,_i \tag{24}$$

where u_i is the displacement vector. Equations (22) through (24) are the governing equations of what is referred to here as a generalized dynamical theory of thermoelasticity.

3. APPLICATION TO ONE-DIMENSIONAL PROBLEMS

The one-dimensional problem, due to its relative simplicity, has had a broad treatment in the literature. The particular problem to be treated here is that of an isotropic homogeneous thermoelastic half-space. The boundary conditions considered are the same as in the wellknown paper by BOLEY and TOLINS (1962), whose results are used for comparison with the present results.

Non-dimensional form of the governing equations

The governing equations can be put in a more convenient form by using non-dimensional variables

$$\left.\begin{aligned} z &= \left(\frac{\lambda + 2\mu}{\rho}\right)^{\frac{1}{2}} \frac{\rho c_E}{k}\,x; \quad \tau = \left(\frac{\lambda + 2\mu}{\rho}\right) \frac{\rho c_E}{k}\,t; \\ \theta &= (T - T_0)/T_0; \quad \Sigma = \sigma/(3\lambda + 2\mu)\,\alpha\,T_0\,; \\ U &= \left[\rho\left(\frac{\lambda + 2\mu}{\rho}\right)^{\frac{1}{2}} \frac{1}{(3\lambda + 2\mu)\,\alpha\,T_0}\,\frac{\rho c_E}{k}\right] u. \end{aligned}\right\} \tag{25}$$

These variables, when substituted into the governing equations for the one-dimensional case, yield the following non-dimensional forms

$$\text{Energy equation}: \ \theta'' - \dot{\theta} - \beta\ddot{\theta} = \tilde{e}\,(\dot{U}' + \beta\ddot{U}') \tag{26}$$

$$\text{Equation of motion}: \ U'' - \ddot{U} = \theta' \tag{27}$$

$$\text{Stress equation}: \ \Sigma = U' - \theta \tag{28}$$

where the primes denote differentiation with respect to z and the dots denote differentiation with respect to τ.

In (26) $\tilde{e}$ represents the wellknown *thermoelastic coupling constant,*

$$\tilde{e} = \frac{(3\lambda + 2\mu)^2\,\alpha^2\,T_0}{(\lambda + 2\mu)\,\rho c_E}, \tag{29}$$

and β represents the non-dimensional form of the relaxation time, which will be called the *relaxation constant,*

$$\beta = \left(\frac{\lambda + 2\mu}{\rho}\right)\left(\frac{\rho c_E}{k}\right)\tau_0. \tag{30}$$

Effect of temperature on the coupling and relaxation constants

It is an experimental fact that both the specific heat c_E and the coefficient of thermal expansion α approach zero as the temperature approaches zero, while the ratio c_E/α approaches a constant which is characteristic of the medium. Consequently, the magnitude of $\tilde{e}$ approaches zero for very low temperatures as is indicated by (29). The energy equation then reduces to

$$\theta'' - \dot{\theta} - \beta\ddot{\theta} = 0, \tag{31}$$

indicating that there is no coupling between thermal and mechanical disturbances at very low temperatures. For slow variations of θ with time, with small values of β, the middle term of the equation dominates and the heat flow is a diffusion type process. However, if the magnitude of β is large, with slow variations in θ, then the spatial variations of θ will proceed as wavelike motions. Consequently, the transport of thermal energy in the medium can be considered to be either a diffusion process or a wavelike process, depending on the magnitude of the relaxation constant β.

The relaxation constant β is expressed in terms of the relaxation time, τ_0, whose order of magnitude can only be determined by considering the mechanism by which thermal energy is carried in the medium. Thermal energy is transported in a solid by two different mechanisms; by quantized electronic excitations which are called free electrons and by the quanta of lattice vibrations which are called phonons. These quanta undergo collisions of a lossful nature, giving rise to thermal resistance in the medium. The relaxation time τ_0 is associated with the average communication time between these collisions for the commencement of resistive flow.

The magnitude of the relaxation time has been estimated for particular types of collision processes. PEIERLS (1955) states that at room temperature the longest collision time occurs for a phonon–electron interaction and is of the order of 10^{-11} sec, while the collision times of phonon–phonon and free electron interactions are both of the order of 10^{-13} sec. However, these times are reduced by imperfections and impurities (e.g. alloying substances) existing in the medium, so that the mean relaxation time τ_0 is not generally known. Consequently, more work is needed to establish the magnitude of τ_0 before the theory presented here can be applied directly to a specific problem. However, something can be said concerning the order of magnitude of τ_0 for certain temperature ranges.

At very low temperatures it is known that the collision time increases. In this temperature range the magnitude of the relaxation constant becomes significant and the energy equation predicts a wave-type phenomenon, 'Second Sound'. OSBORNE (1950), CHESTER (1963) and BROWN *et al.* (1966) use this form of the energy equation (31) to interpret experimental results of heat pulse propagation in Al_2O_3 crystals and liquid helium at very low temperatures. PROHOFSKY and KRUMHANSL (1964) discuss the feasibility of experimental observation of thermal pulses in dielectric materials, indicating the optimum temperature and frequency range for the observation of Second Sound.

The magnitudes of the coupling and relaxation constants were calculated over a range of intermediate and high temperatures by LORD (1966). The values of β, using a representative value of τ_0 equal to 10^{-13} sec, become very small at high temperatures, reducing the governing equations of the generalized theory to the conventional coupled theory of thermoelasticity. Both these theoretically predicted values of the relaxation constant and the success of the coupled theory indicate that 300°K (i.e. room temperature) would be considered 'high temperature.' Furthermore, the product $\tilde{e}\beta$ is much less than either $\tilde{e}$ or β in the intermediate range and at room temperatures. Thus, the $\ddot{U}'$-term can be neglected in the energy equation of the generalized theory, (26).

The possibility of a material which obeys the generalized theory in the intermediate temperature range is now suggested. Although the existence of such a material would need to be verified experimentally, it is still of interest to investigate the effect of the relaxation constant on the behaviour of the material. The solution of the boundary value problem which follows serves this interest.

A problem

Consider an initially quiescent, isotropic, thermoelastic half-space initially at a uniform temperature, $T = T_0$. At time $t = 0$, the free surface $x = 0$ is subjected to a step-strain. This strain is maintained at the surface and, at the same time, the temperature at the free surface is held constant at $T = T_0$. As a result of the mechanical disturbance a strain wave is propagated in the x-direction. This sudden change in strain affects the temperature field, causing thermal disturbances to be propagated in the medium. These two disturbances do not act independently, but are related according to the governing equations derived above.

The initial and boundary conditions may be expressed as

$$\text{Initial conditions}: \ \theta(z, 0) = 0; \quad U(z, 0) = U'(z, 0) = 0 \tag{32}$$

$$\text{Boundary conditions}: \ \theta(0, \tau) = 0; \quad U'(0, \tau) = H(\tau) \tag{33}$$

where $H(\tau)$ represents the Heaviside function. The additional condition of boundedness at infinity is also imposed on θ and U'.

The governing equations for the temperature and displacement fields, (26) and (27), are considered for the case where the strain acceleration term $\ddot{U}'$ is neglected as discussed above. These expressions can be uncoupled, i.e. can be written as two separate differential equations with only one of the dependent variables appearing in each equation. The resulting differential equations are of fourth order

$$\left.\begin{aligned} &\theta'''' - (1 + \beta)\ddot{\theta}'' - (1 + \tilde{e})\dot{\theta}'' + \ddot{\theta} + \beta\dddot{\theta} = 0, \\ &U'''' - (1 + \beta)\ddot{U}'' - (1 + \tilde{e})\dot{U}'' + \ddot{U} + \beta\dddot{U} = 0. \end{aligned}\right\} \tag{34}$$

If (34) are operated on by the Laplace transform, there results the ordinary differential equations in the transform domain

$$\left.\begin{aligned} &\bar{\theta}'''' - p\left[p(1+\beta) + (1+\bar{e})\right]\bar{\theta}'' + p^2(p+\beta p^2)\bar{\theta} = 0, \\ &\bar{U}'''' - p\left[p(1+\beta) + (1+\bar{e})\right]\bar{U}'' + p^2(p+\beta p^2)\bar{U} = 0, \end{aligned}\right\} \tag{35}$$

where $\bar{\theta}(z, p)$ and $\bar{U}(z, p)$ are the Laplace transforms of $\theta(z, \tau)$ and $U(z, \tau)$, respectively, and p is the transform variable. The boundary conditions in the transform domain become

$$\bar{\theta}(0, p) = 0; \quad \bar{U}'(0, p) = 1/p \tag{36}$$

with $\bar{\theta}(z, p)$ and $\bar{U}'(z, p)$ bounded for large z. Equations (35) and (36) can be solved to give the exact solutions for temperature and strain in the transform domain

$$\bar{\theta}(z, p) = \bar{e}\left[\frac{e^{-\alpha_1 z} - e^{-\alpha_2 z}}{\alpha_1^2 - \alpha_2^2}\right] \tag{37}$$

$$\bar{U}'(z, p) = \frac{\left[\alpha_1^2 - (p+\beta p^2)\right]e^{-\alpha_1 z} - \left[\alpha_2^2 - (p+\beta p^2)\right]e^{-\alpha_2 z}}{p(\alpha_1^2 - \alpha_2^2)} \tag{38}$$

where

$$a_{1,2} = \left[\frac{p}{2}\left((1+\beta)p + (1+\bar{e}) \pm \sqrt{\{[(1+\beta)p + (1+\bar{e})]^2 - 4(p+\beta p^2)\}}\right)\right]^{\frac{1}{2}}. \tag{39}$$

Special case

The solutions for temperature and strain, given by (37) and (38), can be inverted and written in closed form if a particular value of β is chosen. These solutions allow certain conclusions to be drawn concerning the general case where β is kept as a parameter.

When the following particular value for the relaxation constant is selected,

$$\beta = 1/(1+\bar{e}), \tag{40}$$

then (39) yields

$$\alpha_1^2 = p^2 + p/\beta; \quad \alpha_2^2 = \beta p^2. \tag{41}$$

These results can be put in (37) and (38) which, with some algebraic manipulation, become

$$\begin{aligned} \bar{U}'(z, p) = \beta\Bigg\{ &\frac{\exp(-\sqrt{(\beta)}\,pz)}{p} \\ &- \frac{\exp[-\sqrt{(\beta)}\,z/\beta(1-\beta)]\exp\{-[p + 1/\beta(1-\beta)]\sqrt{(\beta)}\,z\}}{p + 1/\beta(1-\beta)} \\ &+ \left(\frac{1-\beta}{\beta}\right)\frac{\exp\{-\sqrt{[(p+1/2\beta)^2 - (1/2\beta)^2]}\,z\}}{p} \\ &+ \frac{\exp[-\sqrt{(\{p + 1/\beta(1-\beta) - (\beta+1)/2\beta(1-\beta)\}^2 - (1/2\beta)^2)}\,z]}{p + 1/\beta(1-\beta)}\Bigg\}, \end{aligned} \tag{42}$$

$$\begin{aligned} \bar{\theta}(z, p) = -(1-\beta)\Bigg\{ &\frac{\exp[-\sqrt{(\beta)}\,pz]}{p} \\ &- \frac{\exp[-\sqrt{(\beta)}\,z/\beta(1-\beta)]\exp\{-[p + 1/\beta(1-\beta)]\sqrt{(\beta)}\,z\}}{p + 1/\beta(1-\beta)} \\ &- \frac{\exp\{-\sqrt{[(p+1/2\beta)^2 - (1/2\beta)^2]}\,z\}}{p} \\ &+ \frac{\exp[-\sqrt{(\{p + 1/\beta(1-\beta) - (\beta+1)/2\beta(1-\beta)\}^2 - (1/2\beta)^2)}\,z]}{p + 1/\beta(1-\beta)}\Bigg\}. \end{aligned} \tag{43}$$

Equations (42) and (43) are expressed in terms of known transforms (for example, see ROBERTS and KAUFMAN 1966). Thus, these equations can be inverted directly to give the following closed form solutions for the nondimensional temperature and strain fields in an isotropic, homogeneous, elastic solid

$$U'(z,\tau) = \beta \left\{ F_1(z,\tau) H(\tau - \sqrt{(\beta)}\, z) + \left[F_2(z,t) + \frac{1-\beta}{\beta} F_3(z,\tau) \right] H(\tau - z) \right\}, \quad (44)$$

$$\theta(z,\tau) = -(1-\beta)\{F_1(z,\tau) H(\tau - \sqrt{(\beta)}\, z) + [F_2(z,\tau) - F_3(z,\tau)] H(\tau - z)\}, \quad (45)$$

where

$$F_1(z,\tau) = 1 - \exp[-(\tau - \sqrt{(\beta)}\, z)/\beta(1-\beta)], \quad (46)$$

$$F_2(z,\tau) = \exp[-\tau/\beta(1-\beta)] \left\{ \exp[z(1+\beta)/2\beta(1-\beta)] + \frac{z}{2\beta} \int_z^\tau \exp[t(1+\beta)/2\beta(1-\beta)] \frac{I_1\left\{\frac{1}{2\beta}\sqrt{(t^2-z^2)}\right\}}{\sqrt{(t^2-z^2)}}\, dt \right\}, \quad (47)$$

$$F_3(z,\tau) = \exp[-z/2\beta] + \frac{z}{2\beta} \int_z^\tau \exp[-t/2\beta] \frac{I_1\left\{\frac{1}{2\beta}\sqrt{(t^2-z^2)}\right\}}{\sqrt{(t^2-z^2)}}\, dt, \quad (48)$$

and where $I_1(z)$ is the modified Bessel function of the first kind and of order one, and $H(\tau - k)$ is the Heaviside step-function.

Although the integrals appearing in these equations are not solved explicitly, the expressions are readily amenable to numerical computation. The evaluation of these equations is considered in the following section.

Evaluation of the exact solutions for the special case

Inspection of the solutions (44) and (45) reveals that they consist of two wave fronts propagating in the z-direction at different velocities. The function $H(\tau - \sqrt{(\beta)}\, z)$ represents a wave front travelling in the z-direction with a constant velocity of $1/\sqrt{(\beta)}$, which is the velocity of the thermal disturbance. Hence, this front is called a *thermal wave front*. Similarly, a *strain wave front* is associated with the function $H(\tau - z)$, and travels with a constant velocity equal to unity. Thus, the thermal wave front precedes the strain wave front for this case, since $\beta = 1/(1+\bar{\epsilon}) < 1$.

The magnitude of the strain at the strain wave front can be obtained by setting z equal to τ in (44). Taking cognizance of the properties of the Heaviside function, the magnitude of the strain on either side of the wave front can be determined. Thus, the discontinuity in the strain across the strain wave front is obtained as

$$U'(z,\tau)\,|_{\tau=z+} - U'(z,\tau)\,|_{\tau=z-} = e^{-\tau/2\beta}. \quad (49)$$

This discontinuity travels with the strain wave front, with a speed equal to unity, and its magnitude decays exponentially. Moreover, it can be shown that the total magnitude of the strain at the strain wave front rapidly approaches an asymptotic value given by

$$\lim_{\tau\to\infty} U'(z,\tau)\,|_{z=\tau} = \beta \quad (50)$$

The behaviour of the strain at the thermal wave front is treated in a similar way. Setting z equal to $\tau/\sqrt{\beta}$ in (44), it is seen that the magnitude of the strain vanishes at the thermal wave front

$$U'(z,\tau)\,|_{z=\tau/\sqrt{\beta}} = 0. \quad (51)$$

The behaviour of the temperature can be studied in a similar way to show : (i) there is no 'jump' in temperature across the strain wave front; (ii) the total magnitude of the temperature at the strain wave front rapidly approaches an asymptotic value given by

$$\lim_{\tau\to\infty} \theta(z,\tau)\,|_{z=\tau} = -(1-\beta); \quad (52)$$

(iii) the magnitude of the temperature vanishes at the thermal wave front.

Thus, the strain exhibits a 'jump' at the strain wave front while the temperature is continuous across this front; and both temperature and strain vanish at the thermal wave front.

4. Comparison of Results with Previous Theory

The properties of the solutions for the special case treated here are shown in Fig. 1. The solution at τ_1 corresponds to a short-time solution where the strain 'jump' is still significant, while the solution at τ_2 gives the form of the solution for longer times. The magnitudes of the strain and temperature are given in terms of the coupling constant $\tilde{\epsilon}$ and the relaxation constant β.

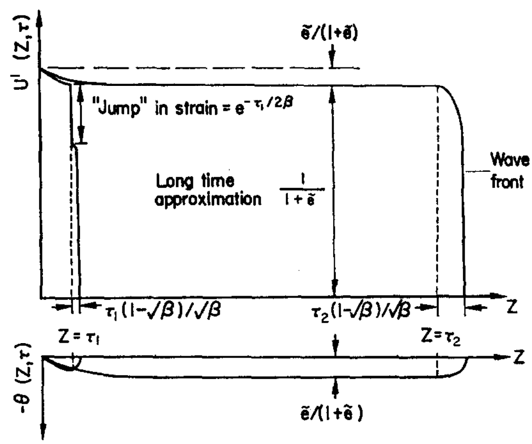

Fig. 1. Form of solution for short and long time, present theory. $\beta = 1/(1 + \tilde{\epsilon})$.

Boley and Tolins' solution, as displayed by Wilms (1964), is shown in Fig. 2. A comparison of the two figures shows that both solutions predict the same asymptotic values for strain and temperature and that there is no 'jump' in temperature at the strain wave front. The 'jump' in strain at the strain wave front is different in the two cases. The fundamental difference in the two solutions, however, is that Boley and Tolins' solution shows a precursory effect extending forward to infinity, while in the present result this effect is replaced by a thermal wave front preceding the strain wave front.

A plot of the computed values of $U'(z, \tau)$ and $\theta(z, \tau)$ is given in Fig. 3. for a coupling coefficient $\tilde{\epsilon}$ equal to 0·03. This value corresponds to 2024S—T4 aluminium at room temperature (Boley and Weiner, 1960). The plot illustrates the rapid decay in the strain 'jump' at the strain wave front. At $\tau = 20$ both strain and temperature have reached their asymptotic magnitudes and the 'jump' in strain has become negligibly small. It should be noted that this value of τ corresponds to $1{\cdot}2 \times 10^{-10}$ sec in real time. Moreover, the strain wave front at $z = \tau = 20$ corresponds to a distance of 2×10^{-5} in. Hence, for the case considered, the strain and the thermal disturbances rapidly attain the form of single wave fronts travelling through the medium with a velocity of $1/\sqrt{\beta}$ in the nondimensional domain.

The actual strain corresponding to a value of U' equal to 0·97 is 0·0134 in./in. for aluminium. Although this magnitude of strain exceeds the yield limit for most

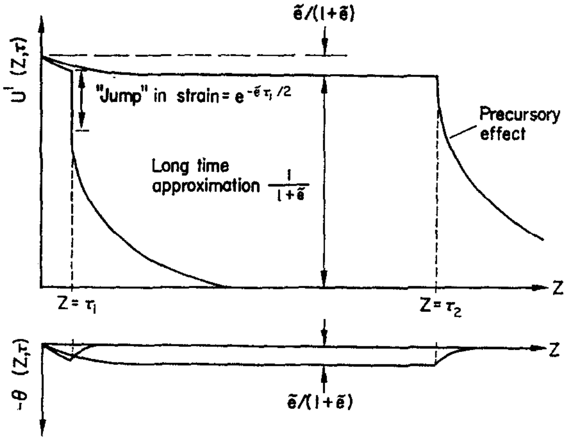

FIG. 2. Form of solution for short and long time, BOLEY and TOLINS (as presented by WILMS).

aluminium alloys, the solution is still representative of the form of solution in the elastic range of the medium. The temperature change associated with this strain is 9°K.

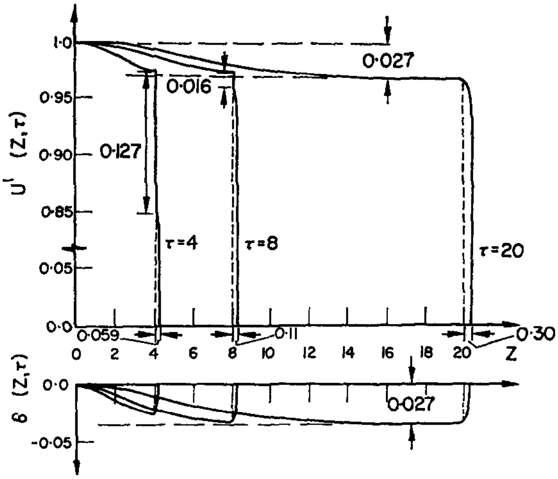

FIG. 3. Variation of strain and temperature with distance at different times. $\beta = 1/(1+\tilde{e})$, $\tilde{e} = 0{\cdot}03$.

5. CONCLUSIONS

The effects of temperature on the relaxation and coupling constants discussed in Section 3 and the form of the exact solution obtained in Section 4, lead to the following conclusions

(i) At very low temperatures the coupling constant $\tilde{e}$ becomes zero. Consequently, the coupling between temperature and strain becomes negligible at temperatures near absolute zero.

(ii) At high temperatures (room temperatures for the materials considered) the relaxation constant β of the generalized theory becomes very small. Thus, at room temperatures the generalized theory reduces to the conventional coupled theory of thermoelasticity.

(iii) The solution obtained by applying the generalized theory (in which the $\dot{U}'$ term is neglected in the energy equation) to a particular boundary value problem is generally similar to that obtained using the conventional coupled theory of thermoelasticity but is different in one important respect, namely, the generalized theory predicts a thermal wave front having a finite propagation velocity. Hence, it is expected that when applied to other problems, the generalized theory should yield solutions not very different in form from those predicted by the conventional coupled theory, except where the propagation of thermal energy is predominant.

(iv) Finally, the generalized theory presented in this work serves to eliminate the paradox of an infinite propagation velocity inherent in the conventional coupled theory of thermoelasticity, and indicates the effect of neglecting higher order terms in the heat conduction equation.

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