PERGAMON International Journal of Engineering Science 40 (2002) 1251–1274

International Journal of Engineering Science

www.elsevier.com/locate/ijengsci

State space approach to two-dimensional generalized thermo-viscoelasticity with two relaxation times

Magdy A. Ezzat *, Mohamed I. Othman, Ahmed M.S. El-Karamany

Department of Mathematics, Faculty of Education, P.O. Box 2801, Salalah-211, Oman

Received 11 June 2001; received in revised form 2 October 2001; accepted 7 November 2001

(Communicated by E.S. ŞUHUBİ)

Abstract

The model of the two-dimensional equations of generalized thermo-viscoelasticity with two relaxation times is established. The state space formulation for two-dimensional problems is introduced. Laplace and Fourier integral transforms are used. The resulting formulation is applied to a problem of a thick plate subject to heating on parts of the upper and lower surfaces of the plate that varies exponentially with time. The Fourier transforms are inverted analytically. A numerical method is employed for the inversion of the Laplace transforms. Numerical results are given and illustrated graphically for the problem considered. Comparisons are made with the results predicted by the coupled theory. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

Since the work of Maxwell, Boltzmann, Voigt, Kelvin and others, the linear viscoelasticity remains an important area of research. Gross [1], Staverman and Schwarzl [2], Alfery and Gurnee [3] and Ferry [4] investigated the mechanical-model representation of linear viscoelastic behavior results. Solution of boundary value problems for linear viscoelastic materials including temperature variations in both quasistatic and dynamic problems made great strides in the last decades, in the work of Biot [5,6], Morland and Lee [7], Tanner [8] and Huilgol and Phan-Thien [9]. Bland [10] linked the solution of linear viscoelasticity problems to corresponding linear elastic solutions.

* Corresponding author. Present address: Department of Mathematics, Faculty of Education, Alexandria of University, Alexandria, Egypt. Tel.: +20-968-299970; fax: +20-968-225620.

E-mail addresses: m_ezzat2000@yahoo.com (M.A. Ezzat), m_i_othman@yahoo.com (M.I. Othman), karamani@omantel.net.om, aelkaramani@yahoo.com (A.M.S. El-Karamany), .

PII: S0020-7225(02)00012-5

Nomenclature

λ, μ	Lame's constants
ρ	density
C_E	specific heat at constant strain
t	time
T	absolute temperature
T_0	reference temperature chosen so that $\vert T - T_0\vert \ll 1$
σ_{ij}	components of stress tensor
ε_{ij}	components of strain tensor
S_{ij}	components of stress deviator tensor
e_{ij}	components of strain deviator tensor
u_i	components of displacement vector
$R(t)$	relaxation function
k	thermal conductivity
A, β, α^*	empirical constants
K	$= \lambda + \frac{2}{3}\mu$ bulk modulus
c_0^2	$= K/\rho$
τ_0, v	relaxation times
α_T	coefficient of linear thermal expansion
γ	$= 3K\alpha_T$
ε	$= \gamma/\rho C_E$
T_0	$= \rho c_0^2 \delta_0/\gamma$
η_0	$= \rho C_E/k$
δ_0	non-dimensional constant
ε_1	$= \delta_0 \varepsilon$

Notable works in this field were the works of Gurtin and Sternberg [11], Sternberg [12] and Ilioushin [13] offered an approximation method for the linear thermal viscoelastic problems. One can refer to the book of Ilioushin and Pobedria [14] for a formulation of the mathematical theory of thermal viscoelasticity and the solutions of some boundary value problems, as well as, to the work of Pobedria [15] for the coupled problems in continuum mechanics. Results of important experiments determining the mechanical properties of viscoelastic materials were involved in the book of Koltunov [16].

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms contrary to the fact that elastic changes produce heat effects. Second, the heat equation is of parabolic type predicting infinite speeds of propagation for heat waves.

Biot [17] formulated the theory of coupled thermoelasticity to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on the temperature. The heat equations for both theories of the diffusion type predict infinite speeds of propagation for heat waves contrary to physical observations. The theory of generalized thermoelasticity with two relaxation times was first introduced by Müller [18]. Green and Laws [19], Green and Lindsay (cf. [20]) then introduced a more explicit version and independently by Şuhubi [21]. In this theory the temperature rates are considered among the constitutive variables. This theory also predicts finite speeds of propagation as in Lord and Shulman's theory of generalized thermoelasticity with one relaxation time [22]. It differs from the latter in that Fourier's law of heat conduction is not violated if the body under consideration has a center of symmetry. Erbay and Şuhubi [23] studied wave propagation in cylinder. Ignaczak [24,25] studied a strong discontinuity wave and obtained a decomposition theorem. Ezzat [26] has also obtained the fundamental solution for this theory. Ezzat and Othman [27] have established the model of two-dimensional equations of generalized magneto-thermoelasticity with two relaxation times in a medium of perfectly conducting medium.

In dealing with generalized or coupled thermoelastic problems the potential function approach is often used. This is not always the most suitable approach. As was discussed in [28], this is mainly due to two reasons. The first is that it is preferable to formulate the problem in terms of the quantities with physical meaning since the boundary and initial conditions of the problem are related directly to these quantities. The second reason is that the solution for a physical problem formulated in natural variables is convergent, while that of a potential function is, unfortunately, not always so. The first writers to introduce the state space formulation in coupled and generalized thermoelasticity were Bahar and Hetnarski [28] and Anwar and Sherief [29], respectively. Ezzat [30] introduces the state space approach to generalized magneto-thermoelasticity with two relaxation times in a medium of perfect conductivity. Ezzat et al. [31] introduce the state space approach to two-dimensional problems of generalized electromagneto-thermoelasticity with two relaxation times. Ezzat et al. [32,33] applied the state space approach to one-dimensional problems of generalized thermo-visco-elasticity.

In the present work we shall formulate the state space approach to two-dimensional problems of thermo-viscoelasticity with two relaxation times in the absence of heat sources. The resulting formulation is applied to a problem of a plate with thermo-isolated surfaces subject to time-dependent compression.

2. Formulation of the problem

We shall consider a thermo-viscoelastic solid occupying the region $-\infty \leqslant x \leqslant \infty$. The governing equations for generalized thermo-viscoelasticity with two relaxation times consist of:

the equation of motion

$$\sigma_{ij,j} = \rho \ddot{u}_i, \tag{1}$$

the generalized heat conduction equation

$$KT_{,ii} = \rho C_{\mathrm{E}} \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) T + \gamma T_0 \dot{e}, \tag{2}$$

the constitutive equation [15,34]

$$S_{ij} = \int_0^t R(t-\tau)\frac{\partial e_{ij}(\bar{x},\tau)}{\partial \tau}\,\mathrm{d}\tau = \hat{R}(e_{ij}) \tag{3}$$

with the assumptions

$$\sigma_{ij}(\bar{x},t) = \frac{\partial \sigma_{ij}(\bar{x},t)}{\partial t} = 0, \quad \varepsilon_{ij}(\bar{x},t) = \frac{\partial \varepsilon_{ij}(\bar{x},t)}{\partial t} = 0, \qquad -\infty < t < 0, \tag{4}$$

where

$$S_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}, \quad e_{ij} = \varepsilon_{ij} - \frac{e}{3}\delta_{ij}, \quad e = \varepsilon_{kk}, \quad \sigma = \frac{\sigma_{kk}}{3}, \quad \sigma_{ij} = \sigma_{ji}, \quad \bar{x} \equiv (x,y,z)$$

and $R(t)$ is the relaxation function which can be taken [16,35] in the form:

$$R(t) = 2\mu\left[1 - A\int_0^t \mathrm{e}^{-\beta t} t^{\alpha^*-1}\,\mathrm{d}t\right], \quad R(0) = 2\mu, \tag{5}$$

where $0 < \alpha^* < 1, \ A > 0, \ \beta > 0$.

Assuming that the relaxation effects of the volume properties of the material are ignored, one can write for the generalized theory of thermo-viscoelasticity with two relaxation times

$$\sigma = K[e - 3\alpha_{\mathrm{T}}(T - T_0 + \nu\dot{T})]. \tag{6}$$

Substituting Eq. (6) into Eq. (3), we obtain

$$\sigma_{ij} = \hat{R}\left(\varepsilon_{ij} - \frac{e}{3}\delta_{ij}\right) + Ke\delta_{ij} - \gamma(T - T_0 + \nu\dot{T})\delta_{ij}. \tag{7}$$

From Eqs. (1) and (7), it follows that

$$\rho\ddot{u}_i = \hat{R}\left(\frac{1}{2}\nabla^2 u_i + \frac{1}{6}e_{,i}\right) + Ke_{,i} - \gamma(T - T_0 + \nu\dot{T})_{,i}. \tag{8}$$

We shall consider only the simplest case of the two-dimensional problem. We assume that all causes producing the wave propagation are independent of the variable z and that waves are propagated only in the xy-plane. Thus all quantities that were appearing in Eqs. (1)–(8) are independent of the variable z. Then the displacement vector has components $(u(x,y,t), v(x,y,t), 0)$ (plane strain problem).

Let us introduce the following non-dimensional variables:

$$x' = c_0\eta_0 x, \quad y' = c_0\eta_0 y, \quad u' = c_0\eta_0 u, \quad v' = c_0\eta_0 v, \quad t' = c_0^2\eta_0 t, \quad \tau_0' = c_0^2\eta_0\tau_0,$$
$$\nu' = c_0^2\eta_0\nu, \quad \theta = \frac{\gamma(T-T_0)}{\rho c_0^2}, \quad R' = \frac{2}{3K}R, \quad \sigma_{ij}' = \frac{\sigma_{ij}}{K}.$$

In terms of these non-dimensional variables, Eqs. (2), (7) and (8), taking the following form (dropping the dashes for convenience):

$$\frac{\partial^2 u}{\partial t^2} = \hat{R}(\varphi) + \frac{\partial e}{\partial x} - \left(\frac{\partial \theta}{\partial x} + v\frac{\partial^2 \theta}{\partial x\,\partial t}\right), \tag{9}$$

$$\frac{\partial^2 v}{\partial t^2} = \hat{R}(\Psi) + \frac{\partial e}{\partial y} - \left(\frac{\partial \theta}{\partial y} + v\frac{\partial^2 \theta}{\partial y\,\partial t}\right), \tag{10}$$

$$\nabla^2\theta = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\theta + \varepsilon\delta_0\frac{\partial e}{\partial t}, \tag{11}$$

$$\sigma_{xx} = \hat{R}\left(\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y}\right) + e - \left(1 + v\frac{\partial}{\partial t}\right)\theta, \tag{12}$$

$$\sigma_{yy} = \hat{R}\left(\frac{\partial v}{\partial y} - \frac{1}{2}\frac{\partial u}{\partial x}\right) + e - \left(1 + v\frac{\partial}{\partial t}\right)\theta, \tag{13}$$

$$\sigma_{zz} = -\frac{1}{2}\hat{R}(e) + e - \left(1 + v\frac{\partial}{\partial t}\right)\theta, \tag{14}$$

$$\sigma_{xy} = \frac{3}{4}\hat{R}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{15}$$

where

$$\varphi = \frac{\partial^2 u}{\partial x^2} + \frac{3}{4}\frac{\partial^2 u}{\partial y^2} + \frac{1}{4}\frac{\partial^2 v}{\partial x\,\partial y}, \tag{16}$$

$$\Psi = \frac{\partial^2 v}{\partial y^2} + \frac{3}{4}\frac{\partial^2 v}{\partial x^2} + \frac{1}{4}\frac{\partial^2 u}{\partial x\,\partial y}, \tag{17}$$

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{18}$$

Taking Laplace transform defined by the relation

$$\bar{f}(x,y,s) = L\{f(x,y,t)\} = \int_0^\infty \mathrm{e}^{-st} f(x,y,s)\,\mathrm{d}t \tag{19}$$

and the Fourier transform

$$\bar{f}^*(q,y,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}qx}\bar{f}(x,y,s)\,\mathrm{d}x \tag{20}$$

on both sides of Eqs. (11)–(17), we obtain

$$s^2\bar{u}^* = s\bar{R}\left(-q^2\bar{u}^* + \frac{3}{4}D^2\bar{u}^* + \frac{\mathrm{i}q}{4}D\bar{v}^*\right) + \mathrm{i}q\bar{e}^* - \mathrm{i}q(1+vs)\bar{\theta}^*, \tag{21}$$

$$s^2\bar{v}^* = s\bar{R}\left(D^2\bar{v}^* - \frac{3}{4}q^2\bar{v}^* + \frac{\mathrm{i}q}{4}D\bar{u}^*\right) + D\bar{e}^* - (1+vs)D\bar{\theta}^*, \tag{22}$$

$$\bar{e}^* = \mathrm{i}q\bar{u}^* + D\bar{v}^*. \tag{23}$$

Eliminating u and v between Eqs. (21), (22) and using (23), we obtain the following equations:

$$s^2\bar{e}^* = (s\bar{R}+1)(D^2-q^2)\bar{e} - (1+vs)(D^2-q^2)\theta, \tag{24}$$

$$(D^2-q^2)\bar{\theta}^* = s(1+\tau_0 s^2)\bar{\theta}^* + \varepsilon_1 s\bar{e}^*, \tag{25}$$

$$\bar{\sigma}^*_{xx} = s\bar{R}\left(\mathrm{i}q\bar{u}^* - \frac{1}{2}D\bar{v}^*\right) - (1+vs)\bar{\theta}^* + \bar{e}^*, \tag{26}$$

$$\bar{\sigma}^*_{yy} = s\bar{R}\left(D\bar{v}^* - \frac{\mathrm{i}q}{2}\bar{u}^*\right) - (1+vs)\bar{\theta}^* + \bar{e}^*, \tag{27}$$

$$\bar{\sigma}^*_{zz} = \left(1 - \frac{1}{2}s\bar{R}\right)\bar{e}^* - (1+vs)\bar{\theta}^*, \tag{28}$$

$$\bar{\sigma}^*_{xy} = \frac{3}{4}s\bar{R}(D\bar{u}^* + \mathrm{i}q\bar{v}^*) \tag{29}$$

and

$$\bar{R} = L\{R(t)\} = \frac{4\mu}{3K}\left[\frac{1}{s} - \frac{A\Gamma(\alpha^*)}{s(s+\beta)^{\alpha^*}}\right], \quad s>0, \tag{30}$$

where $\Gamma(\alpha^*)$ is the gamma function, $L\{\hat{R}(f)\} = s\bar{R}\bar{f}$.

3. State space formulation

We take as state variables in the physical domain the quantities $e, \theta, De, D\theta$. In the transformed domain the state space variables are taken as $\bar{e}^*, \bar{\theta}^*, D\bar{e}^*, D\bar{\theta}^*$ where Eq. (24) together with Eq. (25) gives

$$D^2\bar{e}^* = [\alpha s^2 + q^2 + \alpha\varepsilon_1 s(1+vs)]\bar{e}^* + \alpha s(1+vs)(1+\tau_0 s)\bar{\theta}^*, \tag{31}$$

$$D^2\bar{\theta}^* = [s(1+\tau_0 s) + q^2]\bar{\theta}^* + \varepsilon_1 s\bar{e}^*, \tag{32}$$

where $\alpha = 1/(1+s\bar{R})$.

Eqs. (31) and (32) can be written in matrix form as follows:

$$\frac{\mathrm{d}\tilde{v}(x,y,s)}{\mathrm{d}y} = \tilde{A}(q,s)\tilde{v}(q,y,s), \tag{33}$$

where

$$\tilde{A}(q,s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha s^2 + q^2 + \alpha\varepsilon_1 p & \alpha p(1+\tau_0 s) & 0 & 0 \\ \varepsilon_1 s & p + q^2 & 0 & 0 \end{bmatrix}, \quad \tilde{v}(x,y,s) = \begin{bmatrix} \bar{e}^*(x,y,s) \\ \bar{\theta}^*(x,y,s) \\ D\bar{e}^*(x,y,s) \\ D\bar{\theta}^*(x,y,s) \end{bmatrix}$$

and $p = s(1 + vs)$.

The formal solution of system (33) can be written in the form

$$\tilde{v}(q,y,s) = \exp(y\tilde{A}(q,s))\tilde{v}(q,y_0,s), \tag{34}$$

where y_0 denotes any arbitrary chosen initial value for y.

We shall use the well-known Cayley–Hamilton theorem to find the form of the matrix $\tilde{A}(q,s)$. The characteristic equation of the matrix $\tilde{A}(q,s)$ can be written as

$$k^4 - [\alpha s^2 + 2q^2 + p(1+\alpha\varepsilon_1)]k^2 + [q^4 + q^2(p + \alpha\varepsilon_1 p + \alpha s^2) - \alpha\varepsilon_1 sp(1+\tau_0 s)] = 0. \tag{35}$$

The roots of this equation, namely, k_1^2 and k_2^2, satisfy the relations

$$k_1^2 + k_2^2 = \alpha s^2 + 2q^2 + p(1+\alpha\varepsilon_1), \tag{36}$$

$$k_1^2 k_2^2 = q^4 + q^2[\alpha s^2 + p(1+\alpha\varepsilon_1)] - \alpha\varepsilon_1 sp(1+\tau_0 s). \tag{37}$$

Using Cayley–Hamilton theorem, the finite series representing can be truncated to the following form:

$$\exp(y\tilde{A}(q,s)) = \tilde{L}(q,y,s) = a_0\tilde{I} + a_1\tilde{A} + a_2\tilde{A}^2 + a_3\tilde{A}^3, \tag{38}$$

where $\tilde{I}$ is the unit matrix of order 4 and $a_0, \ldots, a_3$ are some parameters depending on y, q and s.

By Cayley–Hamilton theorem, the characteristic roots $\pm k_1^2$ and $\pm k_2^2$ of the matrix $\tilde{A}$ must satisfy the equations:

$$\begin{aligned}
\exp(k_1 y) &= a_0 + a_1 k_1 + a_2 k_1^2 + a_3 k_1^3, \\
\exp(-k_1 y) &= a_0 - a_1 k_1 + a_2 k_1^2 - a_3 k_1^3, \\
\exp(k_2 y) &= a_0 + a_1 k_2 + a_2 k_2^2 + a_3 k_2^3, \\
\exp(-k_2 y) &= a_0 - a_1 k_2 + a_2 k_2^2 - a_3 k_2^3.
\end{aligned}$$

The solution of the above system is given by

$$
\begin{aligned}
a_0 &= \frac{k_1^2\cosh(k_2y) - k_2^2\cosh(k_1y)}{(k_1^2-k_2^2)},\\
a_1 &= \frac{1}{(k_1^2-k_2^2)}\left[\frac{k_1^2}{k_2}\sinh(k_2y) - \frac{k_2^2}{k_1}\sinh(k_1y)\right],\\
a_2 &= \frac{1}{(k_1^2-k_2^2)}[\cosh(k_1y) - \cosh(k_2y)],\\
a_3 &= \frac{1}{(k_1^2-k_2^2)}\left[\frac{1}{k_1}\sinh(k_1y) - \frac{1}{k_2}\sinh(k_2y)\right].
\end{aligned}
\tag{39}
$$

Substituting expressions (39) into (38) and computing $\tilde{A}^2$ and $\tilde{A}^3$ we obtain, after repeated use of Eqs. (36) and (37), the elements ($\ell_{ij},\ i,j=1,2,3,4$) of the matrix $\tilde{L}(q,y,s)$ as

$$
\begin{aligned}
\ell_{11} &= \frac{1}{(k_1^2-k_2^2)}\left[(k_1^2-q^2-p)\cosh(k_1y) - (k_2^2-q^2-p)\cosh(k_2y)\right],\\
\ell_{12} &= \frac{\alpha p(1+\tau_0 s)}{(k_1^2-k_2^2)}[\cosh(k_1y) - \cosh(k_2y)],\\
\ell_{13} &= \frac{1}{(k_1^2-k_2^2)}\left[\frac{(k_1^2-q^2-p)}{k_1}\sinh(k_1y) - \frac{(k_2^2-q^2-p)}{k_2}\sinh(k_2y)\right],\\
\ell_{14} &= \frac{\alpha p(1+\tau_0 s)}{(k_1^2-k_2^2)}\left[\frac{1}{k_1}\sinh(k_1y) - \frac{1}{k_2}\sinh(k_2y)\right],\\
\ell_{21} &= \frac{\varepsilon_1 s}{(k_1^2-k_2^2)}[\cosh(k_1y) - \cosh(k_2y)],\\
\ell_{22} &= \frac{1}{(k_1^2-k_2^2)}\left[(k_1^2-q^2-p)\cosh(k_2y) - (k_2^2-q^2-p)\cosh(k_1y)\right],\\
\ell_{23} &= \frac{\varepsilon_1 s}{(k_1^2-k_2^2)}\left[\frac{1}{k_1}\sinh(k_1y) - \frac{1}{k_2}\sinh(k_2y)\right],\\
\ell_{24} &= \frac{1}{(k_1^2-k_2^2)}\left[\frac{(k_1^2-q^2-p)}{k_2}\sinh(k_2y) - \frac{(k_2^2-q^2-p)}{k_1}\sinh(k_1y)\right],\\
\ell_{31} &= \frac{1}{(k_1^2-k_2^2)}\left\{\frac{\left[(\alpha s^2+q^2+p\alpha\varepsilon_1)(k_1^2-q^2-p) + p\alpha\varepsilon_1 s(1+\tau_0 s)\right]}{k_1}\sinh(k_1y)\right.\\
&\qquad \left. - \frac{\left[(\alpha s^2+q^2+p\alpha\varepsilon_1)(k_2^2-q^2-p) + \varepsilon_1 s\alpha p(1+\tau_0 s)\right]}{k_2}\sinh(k_2y)\right\},\\
\ell_{32} &= \frac{\alpha p(1+\tau_0 s)}{(k_1^2-k_2^2)}[k_1\sinh(k_1y) - k_2\sinh(k_2y)],\\
\ell_{33} &= \frac{1}{(k_1^2-k_2^2)}\left[(k_1^2-q^2-p)\cosh(k_1y) - (k_2^2-q^2-p)\cosh(k_2y)\right],
\end{aligned}
$$

$$\ell_{34} = \frac{\alpha p(1+\tau_0 s)}{(k_1^2 - k_2^2)}[\cosh(k_1 y) - \cosh(k_2 y)],$$

$$\ell_{41} = \frac{\varepsilon_1 s}{(k_1^2 - k_2^2)}[k_1 \sinh(k_1 y) - k_2 \sinh(k_2 y)],$$

$$\ell_{42} = \frac{1}{(k_1^2 - k_2^2)}\left\{ \frac{\left[(q^2+p)(q^2+p-k_2^2) + \alpha p \varepsilon_1 s(1+\tau_0 s)\right]}{k_1} \sinh(k_1 y) - \frac{\left[(q^2+p)(q^2+p-k_1^2) + \alpha p \varepsilon_1 s(1+\tau_0 s)\right]}{k_2} \sinh(k_2 y) \right\},$$

$$\ell_{43} = \frac{\varepsilon_1 s}{(k_1^2 - k_2^2)}[\cosh(k_1 y) - \cosh(k_2 y)],$$

$$\ell_{44} = \frac{1}{(k_1^2 - k_2^2)}\left[(k_1^2 - q^2 - p)\cosh(k_2 y) - (k_2^2 - q^2 - p)\cosh(k_1 y)\right]. \tag{40}$$

It should be noted here that we have repeatedly used Eqs. (36) and (37) in order to write (40) in the simplest possible form.

4. Application

We consider the problem of a thick plate of finite high $2L$ and of infinite extent with thermal isolated surfaces $y = \pm L$ subjected to time-dependent compression. The initial state of the plate is assumed to be quiescent. The surfaces of the plate are taken to be traction free. Choosing the y-axis perpendicular to the surface of the plate with the origin coinciding with the middle plate, the region Ω under consideration becomes

$$\Omega = \{(x,y,z) : -\infty < x < \infty,\ -L \leqslant y \leqslant L,\ -\infty < z < \infty\}.$$

The boundary conditions of the problem in the transformed domain are

$$\frac{\partial \bar{\theta}^*}{\partial y} = 0 \quad \text{on } y = \pm L, \tag{41}$$

$$\bar{\sigma}^*_{xy} = 0 \quad \text{on } y = \pm L, \tag{42}$$

$$\bar{\sigma}^*_{yy} = -P_0(x,t) \quad \text{on } y = \pm L. \tag{43}$$

We note that due to the symmetry of the problem the temperature θ and the displacement component u are even functions of y, while the displacement component v is an odd function of y. Consequently, the dilatation e is an even function of y.

Let us denote the components of the transformed state vector at the upper surface $y = L$ by $\bar{e}^*_L$, $\bar{\theta}^*_L$, $D\bar{e}^*_L$ and $D\bar{\theta}^*_L$.

Using Eqs. (27) and (29), conditions (41)–(43) reduce to

$$D\bar{e}^*_L = 0, \tag{44}$$

$$D\bar{u}^*_L + \mathrm{i}q\bar{v}^*_L = 0, \tag{45}$$

$$s\bar{R}\left(D\bar{v}^*_L - \frac{\mathrm{i}q}{2}\bar{u}^*_L\right) - (1+vs)\bar{\theta}^*_L + \bar{e}^*_L = -P_0(L,t). \tag{46}$$

The solution of the problem is given by Eq. (33), with y_0 chosen as zero for convenience. Thus, two components of the initial state vector $\tilde{v}_0 = \tilde{v}(q,0,s)$ are known as

$$D\bar{e}^*_0(q,0,s) = D\bar{\theta}^*_0(q,0,s) = 0. \tag{47}$$

The remaining two components $(\bar{e}^*_0(q,0,s), \bar{\theta}^*_0(q,0,s) = 0)$ are obtained from the boundary conditions (44)–(46).

Applying Eq. (33) with $y = L,\ y_0 = 0$ and using Eq. (47), we arrive at

$$\bar{e}^*(q,y,s) = \ell_{11}\bar{e}^*_0 + \ell_{12}\bar{\theta}^*_0, \tag{48}$$

$$\bar{\theta}^*(q,y,s) = \ell_{21}\bar{e}^*_0 + \ell_{22}\bar{\theta}^*_0. \tag{49}$$

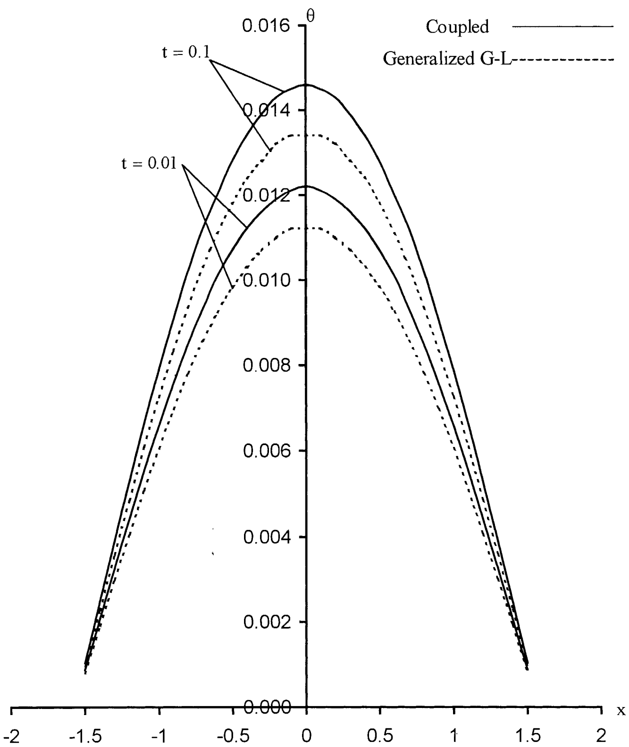

Fig. 1. Temperature distribution on the surface.

In case of symmetry these equations reduce to

$$\bar{e}^*(q,y,s) = \frac{1}{(k_1^2 - k_2^2)} \sum_{i=1}^{2} (-1)^{i-1} \{(k_i^2 - q^2 - p)\bar{e}_\theta^* + \alpha p \bar{\theta}_0^*\} \cosh(k_i y), \tag{50}$$

$$\bar{\theta}^*(q,y,s) = \frac{1}{(k_1^2 - k_2^2)} \sum_{i=1}^{2} (-1)^{i-1} \{\varepsilon_1 p \bar{e}_0^* - (q^2 + \alpha s^2 + p\alpha\varepsilon_1 - k_i^2)\bar{\theta}_0^*\} \cosh(k_i y). \tag{51}$$

Substituting Eq. (23) into Eq. (21), we obtain

$$(D^2 - k_3^2)\bar{u}^* = \mathrm{i}q(\bar{\theta}^* - \alpha_1 \bar{e}^*), \tag{52}$$

where $k_3^2 = \alpha_0 s^2 + q^2$, $\alpha_0 = 4/3s\bar{R}$, $\alpha_1 = \alpha_0 + \frac{1}{3}$.

Substituting Eqs. (50) and (51) into the R.H.S of Eq. (52) and solving the resulting differential equation, we get

$$\bar{u}^* = C\cosh(k_3 y) + \frac{\mathrm{i}q}{(k_1^2 - k_2^2)} \sum_{i=1}^{2} \frac{(-1)^{i-1}}{(k_i^2 - k_3^2)} \times \{[p\varepsilon_1 + \alpha_1(q^2 + p - k_i^2)]\bar{e}_0^* - [\alpha_1 \alpha p + (q^2 + \alpha s^2 + p\alpha\varepsilon_1 - k_i^2)]\bar{\theta}_0^*\} \cosh(k_i y). \tag{53}$$

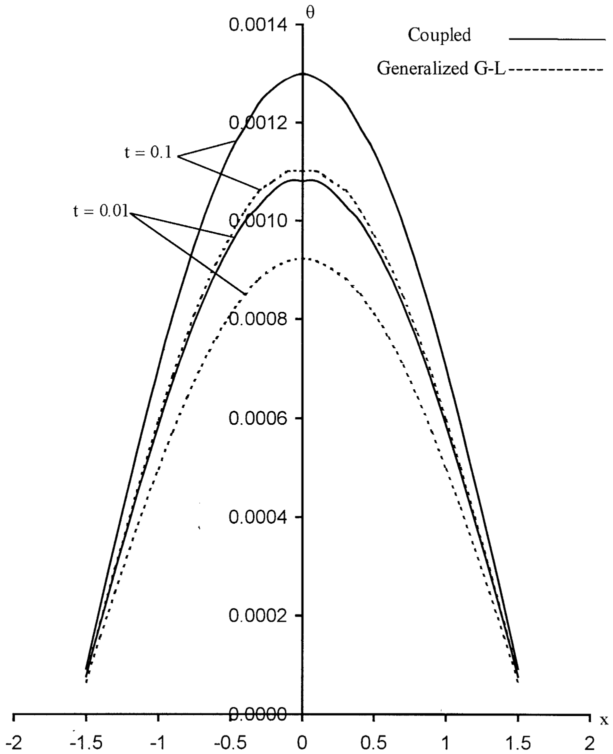

Fig. 2. Temperature distribution on the middle plane.

Substituting Eqs. (51) and (53) into Eq. (29) and integrating the resulting equation, we get

$$\bar{v}^* = \frac{-\mathrm{i}qC}{k_3}\sinh(k_3y) + \frac{1}{(k_1^2-k_2^2)}\sum_{i=1}^{2}\frac{(-1)^{i-1}}{k_i}\{(k_i^2-q^2-p)\bar{e}_0^* + \alpha p)\bar{\theta}_0^*\}\sinh(k_iy)$$
$$+\frac{q^2}{(k_1^2-k_2^2)}\sum_{i=1}^{2}\frac{(-1)^{i-1}}{k_i(k_i^2-k_3^2)}\{[\varepsilon_1\alpha p + \alpha_1(q^2+p-k_i^2)]\bar{e}_0^* + [\alpha_1\alpha p + (q^2+\alpha s^2+p\alpha\varepsilon_1-k_i^2)]\bar{\theta}_0^*\}\sinh(k_iy). \tag{54}$$

The stress components can be obtained by substituting from the above equations into Eqs. (24)–(27). The above approach gives the solution of the problem in the transformed domain in terms of three constants C, $\bar{e}_0^*$ and $\bar{\theta}_0^*$ which can be obtained from the boundary conditions of the articulate problem under consideration.

$$\bar{\sigma}_{xx}^* = \frac{2\mathrm{i}qC}{\alpha_0}\cosh(k_3y) - \frac{2q^2}{\alpha_0(k_1^2-k_2^2)}\sum_{i=1}^{2}\frac{(-1)^{i-1}}{(k_i^2-k_3^2)}$$
$$\times\{[p\varepsilon_1 + (q^2+p-k_i^2)]\bar{e}_0^* - [\alpha_1\alpha p + (q^2+\alpha s^2+p\alpha\varepsilon_1-k_i^2)]\bar{\theta}_0^*\}\cosh(k_iy)$$

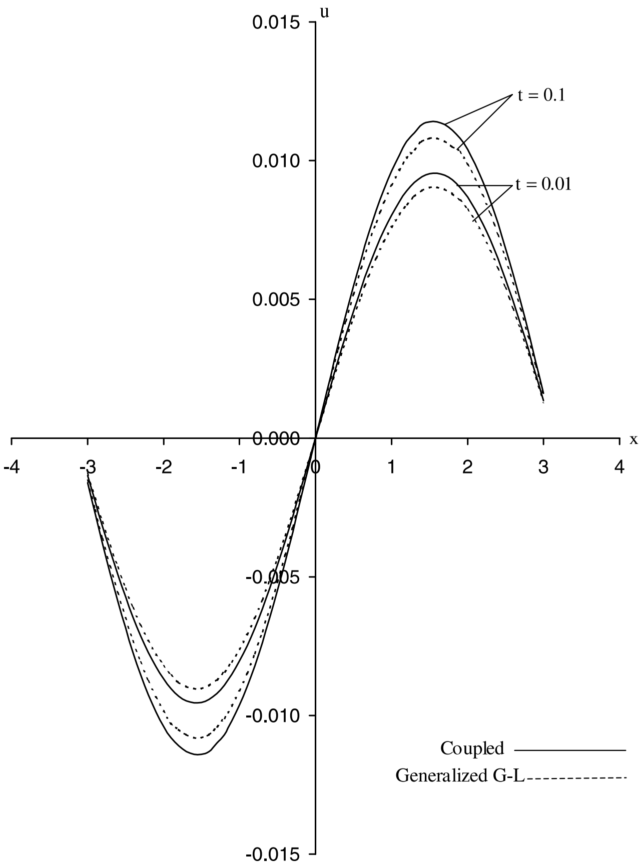

Fig. 3. Horizontal displacement distribution on the surface.

$$+\frac{(3\alpha_0-2)}{3\alpha_0(k_1^2-k_2^2)}\sum_{i=1}^{2}(-1)^{i-1}\{(k_i^2-q^2-p)\bar{e}_0^*+\alpha p\bar{\theta}_0^*\}\cosh(k_iy)$$

$$-\frac{1}{(k_1^2-k_2^2)}\sum_{i=1}^{2}(-1)^{i-1}\{p\varepsilon_1\bar{e}_0^*-\left(q^2+\alpha s^2+p\alpha\varepsilon_1-k_i^2\right)\bar{\theta}_0^*\}\cosh(k_iy), \tag{55}$$

$$\bar{\sigma}_{yy}^*=-\frac{2iqC}{\alpha_0}\cosh(k_3y)-\frac{2q^2}{\alpha_0(k_1^2-k_2^2)}\sum_{i=1}^{2}\frac{(-1)^{i-1}}{(k_i^2-k_3^2)}$$

$$\times\{[p\varepsilon_1+\alpha_1(q^2+p-k_i^2)]\bar{e}_0^*-[\alpha_1\alpha p+(q^2+\alpha s^2+p\alpha\alpha_1-k_i^2)]\bar{\theta}_0^*\}\cosh(k_iy)$$

$$+\frac{1}{\alpha(k_1^2-k_2^2)}\sum_{i=1}^{2}(-1)^{i-1}\{(k_i^2-q^2-p)\bar{e}_0^*+\alpha p\bar{\theta}_0^*\}\cosh(k_iy)$$

$$-\frac{1}{(\mathrm{k}_1^2-k_2^2)}\sum_{i=1}^{2}(-1)^{i-1}\{p\varepsilon_1\bar{e}_0^*-(q^2+\alpha s^2+p\alpha\varepsilon_1-k_i^2)\bar{\theta}_0^*\}\cosh(k_iy), \tag{56}$$

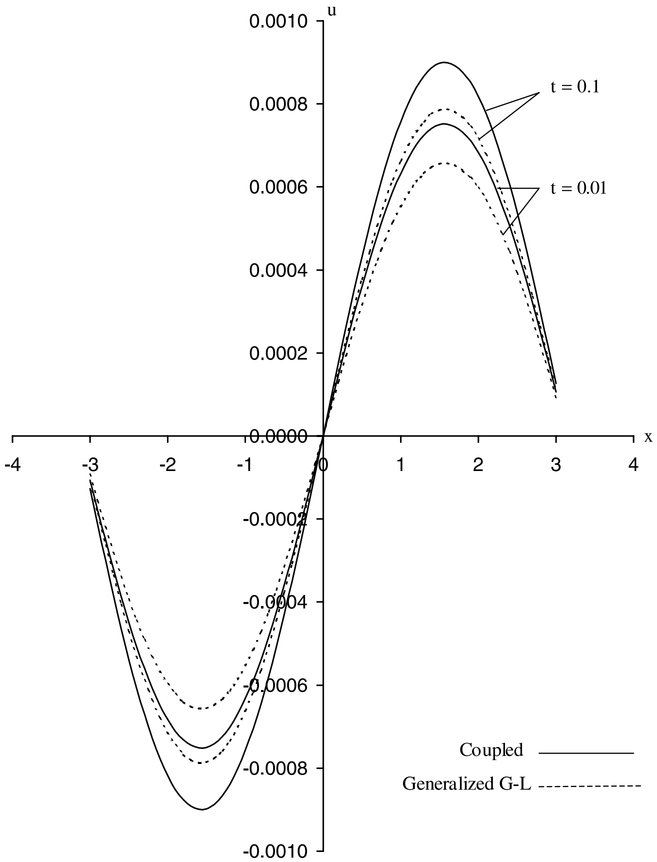

Fig. 4. Horizontal displacement distribution on the middle plane.

$$\bar{\sigma}^*_{zz} = \frac{(2 - s\bar{R})}{2(k_1^2 - k_2^2)} \sum_{i=1}^{2} (-1)^{i-1} \{[(k_i^2 - q^2 - p)\bar{e}^*_0 + p\alpha\bar{\theta}^*_0]\} \cosh(k_i y)$$
$$- \frac{1}{(k_1^2 - k_2^2)} \sum_{i=1}^{2} (-1)^{i-1} \{p\varepsilon_1 \bar{e}^*_0 - (q^2 + \alpha s^2 + p\alpha\varepsilon_1 - k_i^2)\bar{\theta}^*_0\} \cosh(k_i y), \quad (57)$$

$$\bar{\sigma}^*_{xy} = \frac{(q^2 + k_3^2)}{\alpha_0 k_3} C \sinh(k_3 y) + \frac{\mathrm{i}q(1 - q^2)}{\alpha_0 (k_1^2 - k_2^2)} \sum_{i=1}^{2} \frac{(-1)^{i-1}}{k_i (k_i^2 - k_3^2)}$$
$$\times \{[p\varepsilon_1 + \alpha_1 (q^2 + p - k_i^2)]\bar{e}^*_0 - [\alpha_1 \alpha p (q^2 + \alpha s^2 + p\alpha\varepsilon_1 - k_i^2)]\bar{e}^*_0\} \sinh(k_i y)$$
$$+ \frac{\mathrm{i}q}{\alpha_0 (k_1^2 - k_2^2)} \sum_{i=1}^{2} (-1)^{i-1} \{(k_i^2 - q^2 - p)\bar{e}^*_0 + \alpha p \bar{\theta}^*_0\} \sinh(k_i y), \quad (58)$$

where

$$\bar{e}^*_0 = \frac{P^*_0}{\Delta} [B_{11} B_{16} - B_{13} B_{14}],$$

$$\bar{\theta}^*_0 = \frac{P^*_0(q,s)}{\Delta} [B_{12} B_{14} - B_{11} B_{15}],$$

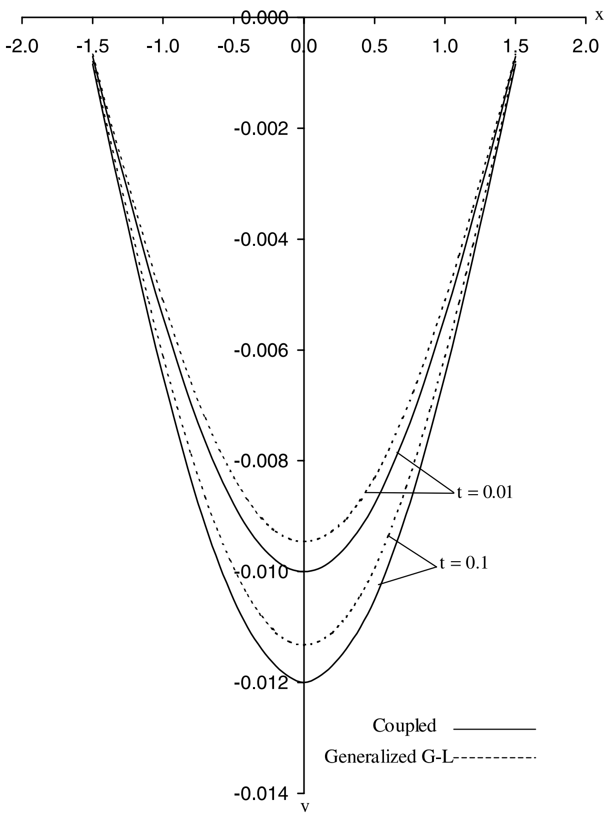

Fig. 5. Vertical displacement distribution on the surface.

$$C = \frac{P_0^*(q,s)}{\Delta}[B_{13}B_{15} - B_{12}B_{16}],$$

$$B_{11} = \frac{(k_3^2 + q^2)}{k_3}\sinh(k_3L),$$

$$B_{12} = \frac{\mathrm{i}q}{(k_1^2 - k_2^2)}\sum_{i=1}^{2}\frac{(-1)^{i-1}}{k_i(k_i^2 - k_3^2)}$$
$$\times\{(q^2 + k_i^2)[p\varepsilon_1 + \alpha_1(q^2 + p - k_i^2)\sinh(k_iL)] + (k_i^2 - k_3^2)(k_i^2 - q^2 - p)\}\sinh(k_iL),$$

$$B_{13} = \frac{\mathrm{i}q}{(k_1^2 - k_2^2)}\sum_{i=1}^{2}\frac{(-1)^{i-1}}{k_i(k_i^2 - k_3^2)}\{(q^2 + k_i^2)[\alpha_1\alpha p(1 + \tau_0 s) + (1 + vs)(k_i^2 - q^2 - \alpha s^2 - p\alpha\varepsilon_1)]$$
$$+ \alpha p(1 + \tau_0 s)(k_i^2 - k_3^2)\}\sinh(k_iL),$$

$$B_{14} = -\frac{2\mathrm{i}q}{\alpha_0}\cosh(k_3L),$$

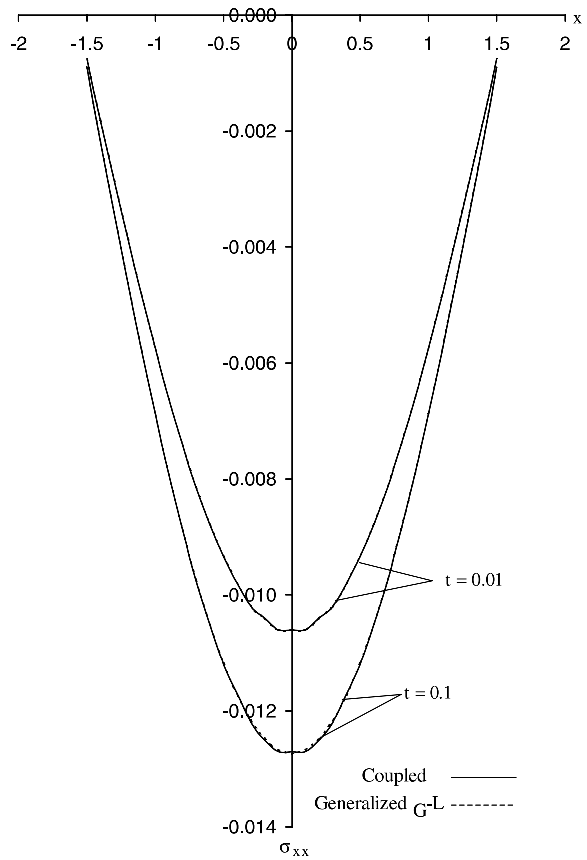

Fig. 6. Stress component σ_{xx} distribution on the surface.

$$B_{15} = \frac{1}{\alpha_0(k_1^2 - k_2^2)} \sum_{i=1}^{2} (-1)^{i-1} \times \left\{ (k_i^2 - q^2 - p) + \frac{2q^2}{(k_i^2 - k_3^2)} [p\varepsilon_1 + \alpha_1(q^2 + p - k_i^2)] - \alpha_0 \varepsilon_1 s \right\} \cosh(k_i L),$$

$$B_{16} = \frac{1}{\alpha_0(k_1^2 - k_2^2)} \sum_{i=1}^{2} (-1)^{i-1} \times \left\{ \alpha p(1 + \tau_0 s) - \frac{2q^2}{(k_i^2 - k_3^2)} [\alpha_1 \alpha p(1 + \tau_0 s) + (1 + vs)(k_i^2 - q^2 - \alpha s^2 - p\alpha\varepsilon_1)] - \alpha_0(k_i^2 - q^2 - \alpha s^2 - p\alpha\varepsilon_1) \right\} \cosh(k_i L),$$

$$B_{17} = \varepsilon_1 s \sum_{i=1}^{2} (-1)^{i-1} k_i \sinh(k_i L),$$

$$B_{18} = \sum_{i=1}^{2} (-1)^{i-1} k_i [(k_i^2 - q^2 - \alpha s^2 - p\alpha\varepsilon_1)] \sinh(k_i L),$$

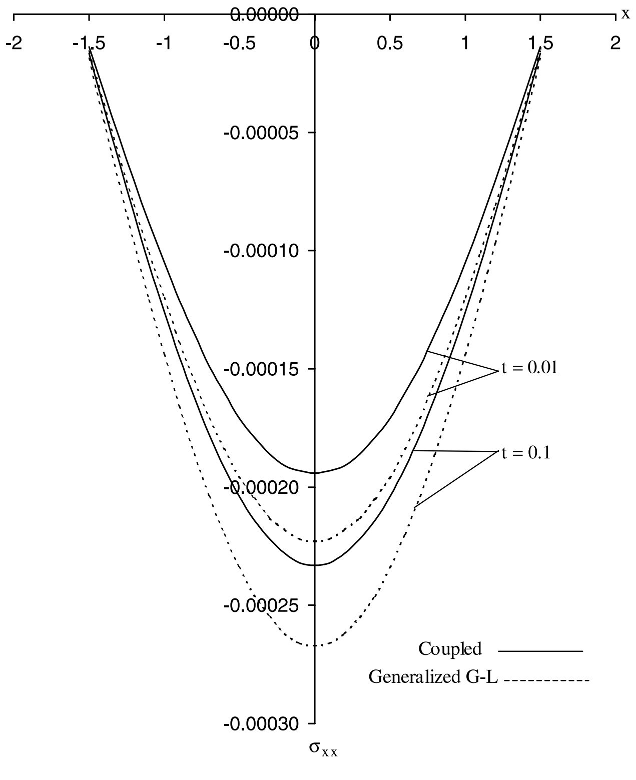

Fig. 7. Stress component σ_{xx} distribution on the middle plane.

$$\Delta = B_{11}(B_{15}B_{18} - B_{16}B_{17}) - B_{14}(B_{12}B_{18} - B_{13}B_{17}).$$

This completes the solution of the problem in the transformed domain.

5. Inversion of the transforms

In order to obtain the solution of the problem in the physical domain, we have to invert the iterated transforms in Eqs. (50), (53) and (54). These expressions can be formally expressed as function of y and the parameters of the Fourier and Laplace transforms q and s, of the form $\bar{f}^*(q,y,s)$. First, we invert the Fourier transform using the inversion formula given previously.

This gives the Laplace transform expression $\bar{f}(q,y,s)$ of the function $f(q,y,t)$ as

$$\bar{f}(x,y,s), = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}qx} f(q,y,s)\,\mathrm{d}q = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \{f_\mathrm{e} \cos qx + \mathrm{i} f_\mathrm{o} \sin qx\}\,\mathrm{d}q, \tag{59}$$

where f_e and f_o denote the even and odd parts of the function $\bar{f}^*(q,y,s)$, respectively.

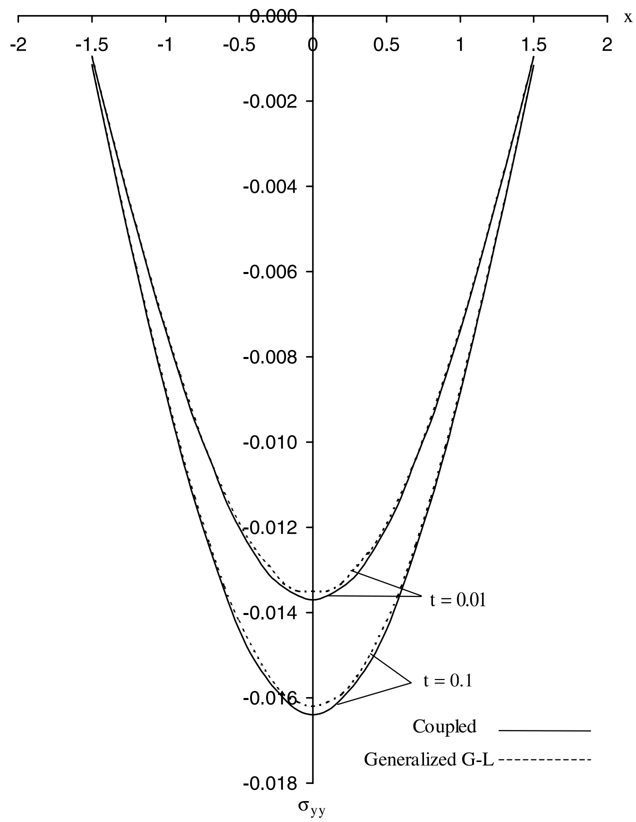

Fig. 8. Stress component σ_{yy} distribution on the surface.

We shall now outline the numerical inversion method used to find the solution in the physical domain. For fixed values of q, x and y the function inside braces in the last integral can be considered as a Laplace transform $\bar{g}(s)$ of some function $g(t)$. The inversion formula for the Laplace transforms can be written as

$$g(t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{st} \bar{g}(s)\, \mathrm{d}s,$$

where c is an arbitrary real number greater than all the real parts of the singularities of $\bar{g}(s)$. Taking $s = c + \mathrm{i}y$, the above integral takes the form

$$g(t) = \frac{\mathrm{e}^{ct}}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ty}\, \bar{g}(c + \mathrm{i}y)\, \mathrm{d}y.$$

Expanding the function $h(t) = \exp(-ct)\, g(t)$ in a Fourier series in the interval $[0, 2L]$, we obtain the approximate formula [36]

$$g(t) = g_\infty(t) + E_\mathrm{D},$$

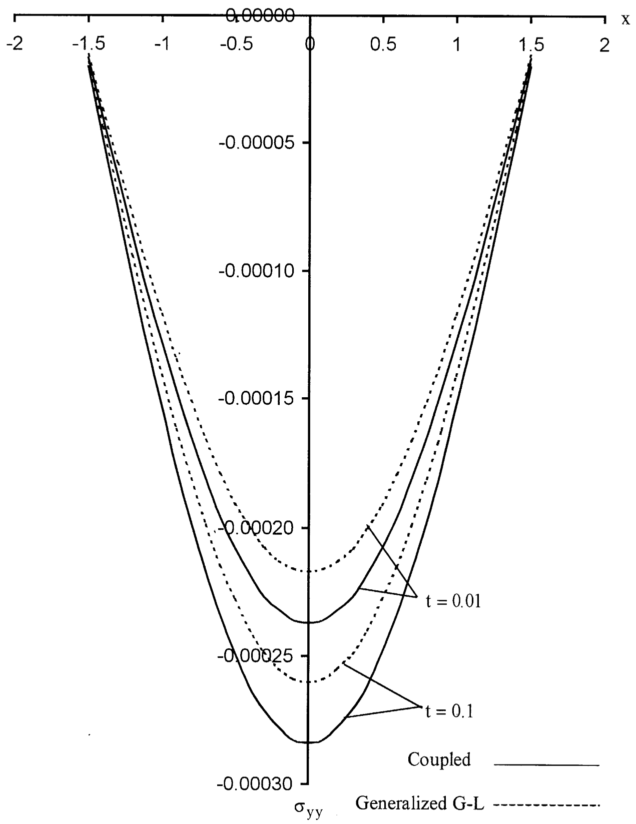

Fig. 9. Stress component σ_{yy} distribution on the middle plane.

where

$$g_{\infty}(t) = \frac{1}{2}C_0 + \sum_{k=1}^{\infty} C_k \quad \text{for } 0 \leqslant t \leqslant 2L \tag{60}$$

and

$$C_k = \frac{e^{ct}}{L}\mathrm{Re}[e^{ik\pi t/L}\bar{g}(c + ik\pi/L)]. \tag{61}$$

E_D, the discretization error, can be made arbitrary small [37].

As value of $g(t)$ becomes the infinite series in (60) it can be summed up to a finite number N of terms, the approximate

$$g_N(t) = \frac{1}{2}C_0 + \sum_{k=1}^{N} C_k \quad \text{for } 0 \leqslant t \leqslant 2L. \tag{62}$$

Using the above formula to evaluate $g(t)$, we introduce a truncation error E_T which must be added to the discretization error to produce the total approximation error.

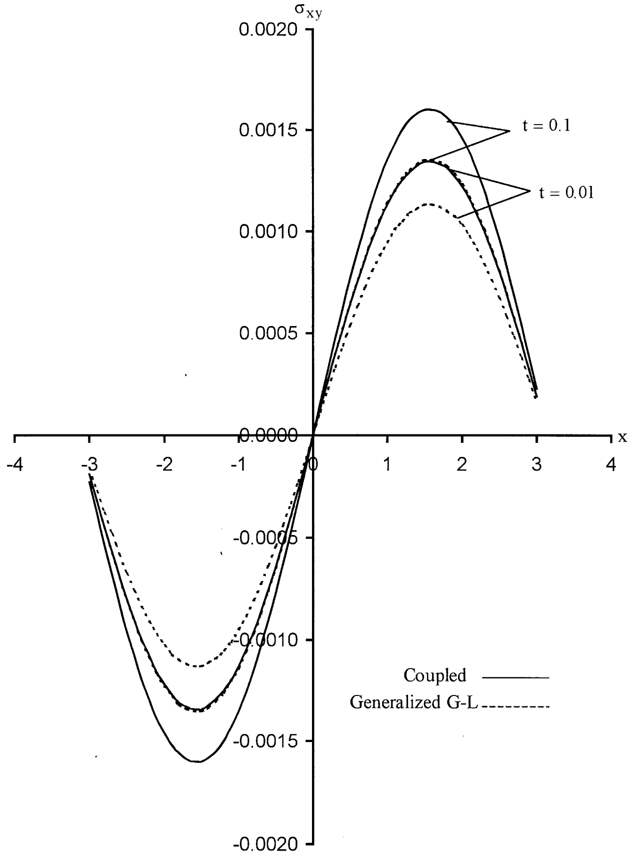

Fig. 10. Stress component σ_{xy} distribution on the surface.

Two methods are used to reduce the total error. First, the Korrecktur-method is used to reduce the discretization error. Next, the ε-algorithm is used to reduce the truncation error and hence to accelerate convergence.

The Korrecktur-method uses the following formula to evaluate the function $g(t)$:

$$g(t) = g_{\infty}(t) - e^{-2cL} g_{\infty}(2L + t) + E'_{D},$$

where the discretization error $|E'_{D}| \ll |E_{D}|$ as in [37]. Thus, the approximate value of $g(t)$ becomes

$$g_{NK}(t) = g_{N}(t) - e^{-2cL} g_{N'}(2L + t), \tag{63}$$

where N' is an integer such that $N' < N$.

We shall now describe the ε-algorithm, which is used to accelerate the convergence of the series in (62). Let N be an odd natural number and let $S_m = \sum_{k=1}^{m} c_k$ be the sequence of partial sums of (62). We define ε-sequence by $\varepsilon_{0,m} = 0$, and $\varepsilon_{1,m} = s_m$ and

$$\varepsilon_{n+1,m} = \varepsilon_{n-1,m+1} + \frac{1}{\varepsilon_{n,m+1} - \varepsilon_{n,m}}, \quad n, m = 1, 2, 3, \ldots \tag{64}$$

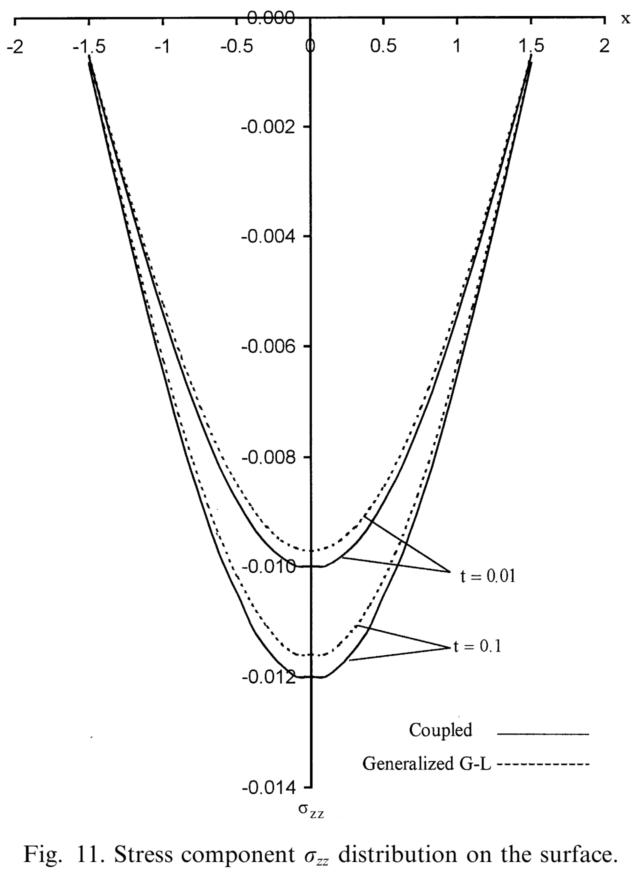

Fig. 11. Stress component σ_{zz} distribution on the surface.

It can be shown in [38] that the sequence $\varepsilon_{1,1}, \varepsilon_{3,1}, \ldots, \varepsilon_{N,1}$ converges to $g(t) + E_D - \frac{1}{2}c_0$ faster than the sequence of partial sums $s_m,\ m = 1, 2, \ldots$

The actual procedure used to invert the Laplace transforms consists of using Eq. (63) together with the ε-algorithm. The values of c and L are chosen according the criteria outlined in [37].

The last step in the inversion is to evaluate the integral in (59). This was done using Romberg integration with adaptive step size. This method uses the results from successive refinements of the extended trapezoidal rule followed by extrapolation of the results to the limit when the step size tends to zero. The details can be found in [38].

6. Numerical results

In order to illustrate the above results graphically the source $r(x, t)$ was taken in the following form:

$$r(x, t) = H(x - |a|)H(t)\exp(-bt),$$

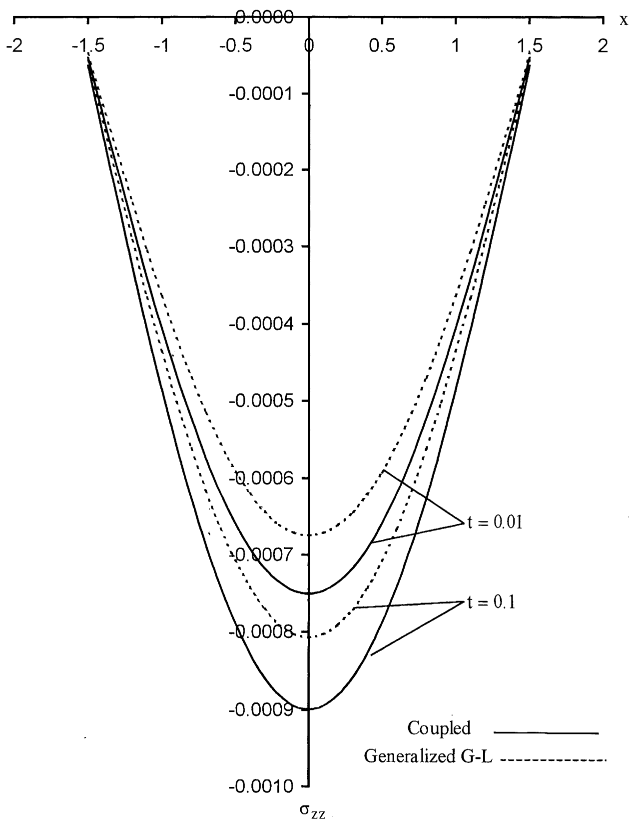

Fig. 12. Stress component σ_{zz} distribution on the middle plane.

where a and b are fixed constants and H denotes Heaviside unit step function. This represents a localized heat source acting in the region $-L \leqslant x \leqslant L$ starting at $t = 0$ with a value of unity and exponentially decaying in time. The double transform of $r(x, t)$ is given by

$$\bar{r}^*(q, s) = \sqrt{\frac{2}{\pi}} \frac{\sin(qa)[1 + iq\pi\delta(q)]}{q(s + b)},$$

where $\delta(q)$ denotes the Dirac delta function.

As a numerical example we have considered polymethyl methacrylate which has wide applications in industry and medicine.

The numerical constants are taken as

$$\frac{4\mu}{3K} = 0.8, \quad A = 0.106, \quad \varepsilon_1 = 0.045, \quad \beta = 0.005, \quad T_0 = 773 \text{ K}, \quad L = 2, \quad \alpha^* = 0.5,$$
$$\alpha = 0.59037.$$

The numerical technique outlined above was used to invert the iterated transforms in Eq. (51) giving the temperature and Eqs. (53)–(58) giving the displacement components u, v and stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}$ on the surface of plane $y = L$ and on the middle plane $y = 0$. The results are shown in Figs. 1–12, respectively. In these figures, the solid and dotted lines represent the solution obtained at two values of time, namely $t = 0.01$ and $t = 0.1$, respectively. The graph shows the four curves predicted by the different theories of thermoelasticity, coupled thermoelasticity ($\tau_0 = 0.0, v = 0.0$) and Green and Lindsay theory ($\tau_0 = 0.02, v = 0.03$). We note that since the displacement component v and σ_{xy} is an odd function of y, its value on middle plane is always zero.

We note that results for the all functions for generalized theory are distinctly different from those obtained for the coupled theory. This due to the fact that thermal waves in the coupled theory travel with an infinite speed of propagation as opposed to finite speed in the generalized case. It is clear that for small values of time the solution is localized in a finite region. This region grows with increasing time and its edge is the location of the wave front. This region is determined by the values of time t and relaxation times τ_0 and v.

7. Concluding remarks

The importance of state space analysis is recognized in fields where the time behavior of physical process is of interest [28].

The state space approach is more general than the classical Laplace and Fourier transform techniques. Consequently, state space is applicable to all systems that can be analyzed by integral transforms in time, and is applicable to many systems for which transform breaks down [36].

Owing to the complicated nature of the governing equations for the generalized thermo-viscoelasticity with one relaxation time, few attempts have been made to solve problems in this field. These attempts utilized approximate method valid for only a specific range of some parameters.

In this work, the method of direct integration was by means of the matrix exponential, which is a standard approach in modern control theory and developed in detail in many texts [39,40].

The method used in the present work is applicable to a wide range of problems. It can be applied to problems, which are described by the linearized Navier–Stokes equations in hydrodynamic theory [41].

References

[1] B. Gross, Mathematical Structure of the Theories of Viscoelasticity, Hemann, Paris, 1953.
[2] A.J. Staverman, F. Schwarzl, in: H.A. Stuart (Ed.), Die Physik der Hochpolymeren, vol. 4, Springer, Berlin, 1956 (Chapter 1).
[3] T. Alfrey, E.F. Gurnee, in: F.R. Eirich (Ed.), Rheology Theory and Applications, vol. 1, Academic Press, New York, 1956.
[4] J.D. Ferry, Viscoelastic Properties of Polymers, Wiley, New York, 1970.
[5] M.A. Biot, Theory of stress–strain relations in an isotropic viscoelasticity and relaxation phenomena, J. Appl. Phys. 25 (1954).
[6] M.A. Biot, Variational principal in irreversible thermodynamics with application to viscoelasticity, Phys. Rev. 97 (1955).
[7] L.W. Morland, E.H. Lee, Trans. Soc. Rheol. 4 (1960).
[8] R.I. Tanner, Engineering Rheology, Oxford University Press, Oxford, 1988.
[9] R. Huilgol, N. Phan-Thien, Fluid Mechanics of Viscoelasticity, Elsevier, Amsterdam, 1997.
[10] D.R. Bland, The Theory of Linear Viscoelasticity, Pergamon Press, Oxford, 1960.
[11] M.E. Gurtin, E. Sternberg, On the linear theory of viscoelasticity, Arch. Rational Mech. Anal. 11 (1962).
[12] E. Sternberg, On the analysis of thermal stresses in viscoelastic solids, Brown Univ. Dir, Appl. Math. TR 19 (1963).
[13] A.A. Ilioushin, The approximation method of calculating the constructures by linear thermal viscoelastic theory, Mekh. Polimerov, Riga 2 (1968).
[14] A.A. Ilioushin, B.E. Pobedria, Mathematical Theory of Thermal Visco-elasticity, Nauka, Moscow, 1970.
[15] B.E. Pobedria, Coupled problems in continuum mechanics, J. Durabilility and Plasticity (1) (1984).
[16] M.A. Koltunov, Creeping and Relaxation, Moscow, 1976.
[17] M. Biot, Thermoelasticity and irreversible thermodynamics, J. Appl. Phys. 27 (1956) 240.
[18] I. Müller, Arch. Rational Mech. Anal. 41 (1976) 319–332.
[19] A.E. Green, N. Laws, Arch. Rational Mech. Anal. 45 (1972) 47–53.
[20] A.E. Green, N. Laws, Elasticity 45 (1972) 1–7.
[21] E.S. Şuhubi, Thermoelastic solids, in: A.C. Eringen (Ed.), Continuum Physics, vol. II, Academic Press, New York, 1975 (Chapter 2).
[22] H. Lord, Y. Shulman, A generalized dynamical theory of thermoelasticity, J. Mech. Phys. Solid 15 (1967) 299–309.
[23] S. Erbay, E.S. Şuhubi, Longitudinal wave propagation in a generalized thermo-elastic cylinder, J. Thermal Stresses 90 (1986) 279–295.
[24] J. Ignaczak, A strong discontinuity wave in thermoelastic with relaxation times, J. Thermal Stresses 8 (1985) 25–40.
[25] J. Ignaczak, Decomposition theorem for thermoelasticity with finite wave speeds, J. Thermal Stresses 1 (1978) 41–52.
[26] M.A. Ezzat, Fundamental solution in thermoelasticity with two relaxation times, Internat. J. Engrg. Sci. 33 (1995) 2011–2020.
[27] M.A. Ezzat, M.I. Othman, Electromagneto-thermoelastic plane waves with two relaxation times in a medium of perfect conductivity, Internat. J. Engrg. Sci. 38 (2000) 107–120.
[28] L. Bahar, R. Hetnarski, State space approach to thermoelasticity, J. Thermal Stresses 1 (1978) 135.
[29] M. Anwar, H.H. Sherief, State space approaches to generalized thermo-elasticity, J. Thermal Stresses 11 (1988) 353.

[30] M.A. Ezzat, State space approach to generalized magneto-thermoelasticity with two relaxation times in a medium of perfect conductivity, Internat. J. Engrg. Sci. 35 (1997) 741–752.

[31] M.A. Ezzat, M.I. Othman, A.A. Smaan, State space approach to two-dimensional electromagneto-thermoelastic problem with two relaxation times, Internat. J. Engrg. Sci. 39 (2001) 1383–1404.

[32] M.A. Ezzat, M.I. Othman, A.S. El Karamany, State space approach to generalized thermo-viscoelasticity with two relaxation times, Internat. J. Engrg. Sci. 40 (2002) 283–302.

[33] M.A. Ezzat, A.S. El Karamany, A.A. Smaan, State space formulation to generalized thermo-viscoelasticity with thermal relaxation, J. Thermal Stresses 24 (2001) 823–846.

[34] Y.C. Fung, Foundation of Solid Mechanics, Prentice-Hall, Englewood Cliffs, NJ, 1968.

[35] A.S. El Karamany, Deformation of a non-homogeneous viscoelastic hollow sphere, Engrg. Trans. Warsaw 31 (1983) 267.

[36] D. Wibery, Theory and Problems of State Space and Linearized System: Schaum's Outline Series Engineering, MC Graw-Hill Bodes, New York, 1971.

[37] G. Honig, U. Hirdes, A method for the numerical inversion of the Laplace transform, J. Comput. Appl. Math. 10 (1984) 113–132.

[38] H. William, Press al. Numerical Recipes, Cambridge University Press, Cambridge, 1986.

[39] K. Ogata, State Space Analysis Control System, Prentice-Hall, Englewood Cliffs, NJ, 1967 (Chapter 6).

[40] M.A. Ezzat, State space approach to unsteady two-dimensional free convection flow through a pours medium, Canad. J. Phys. 72 (1994) 311–317.

[41] M.A. Ezzat, Free convection effects on perfectly conducting fluid, Internat. J. Engrg. Sci. 39 (2001) 799–819.