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Generalized thermo-viscoelastic plane waves with two relaxation times

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Abstract

The model of the two-dimensional generalized thermo-viscoelasticity with two relaxation times (Green and Lindsay theory) is established. The normal mode analysis is used to obtain the exact expressions for the temperature distribution, thermal stresses and the displacement components. The resulting formulation is applied to three different concrete problems. The first deals with a thick plate subjected to a time-dependent heat source on each face. The second concerns to the case of a heated punch moving across the surface of a semi-infinite thermo-viscoelastic half-space subjected to appropriate boundary conditions and the third problem deals with a plate with thermo-isolated surfaces subjected to a time-dependent compression. Numerical results are given and illustrated for each problem. Comparisons are made with the results predicted by the coupled theory. © 2002 Published by Elsevier Science Ltd.

1. Introduction

Since the work of Maxwell, Boltzmann, Voigt, Kelvin and others, the linear viscoelasticity remains an important area of research. Gross [1], Staverman and Schwarzl [2], Alfery and Gurnee [3] and Ferry [4] investigated the mechanical-model representation of linear viscoelastic behavior results. Solution of boundary value problems for linear viscoelastic materials including temperature variations in both quasistatic and dynamic problems made great strides in the last decades,

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Nomenclature

λ, μ	Lame's constants
K	$\lambda + \frac{2}{3}\mu$
ρ	density
C_E	specific heat at constant strain
t	time
T	absolute temperature
T_0	reference temperature chosen so that $\lvert T - T_0 \rvert \ll 1$
u_i	components of displacement vector
ε_{ij}	components of strain tensor
e	ε_{kk} the dilatation
σ_{ij}	components of stress deviator
e_{ij}	components of strain deviator
k	thermal conductivity
τ_0, v	two relaxation times
α_t	coefficient of linear thermal expansion
γ	$3K\alpha_t$
ε	$\gamma/(\rho C_E)$
η_0	$(\rho C_E)/k$
c_0^2	K/ρ
ε_1	$\delta_0\varepsilon$
T_0	$(\delta_0\rho c_0^2)/\gamma = \delta_0/3\alpha_T$
δ_0	non-dimensional number
α^*, β, A	empirical constants

in the works of Biot [5,6], Morland and Lee [7], Tanner [8] and Huilgol and Phan-Thien [9]. Bland [10] linked the solution of linear-viscoelasticity problems to corresponding linear elastic solutions. Notable works in this field were the works of Gurtin and Sternberg [11], Sternberg [12] and Ilioushin [13] offered an approximation method for the linear thermal viscoelastic problems. One can refer to the book of Ilioushin and Pobedria [14] for a formulation of the mathematical theory of thermal viscoelasticity and the solutions of some boundary value problems, as well as, to the work of Pobedria [15] for the coupled problems in continuum mechanics. Results of important experiments determining the mechanical properties of viscoelastic materials were involved in the book of Koltunov [16].

Two generalizations to the coupled theory are introduced. The first is due to Lord and Shulman [17] who obtained a wave-type heat equation by postulating a new law of heat conduction to replace the classical Fourier's law. This new law contains the heat flux vector as well as its time derivative. It contains also a new constant that acts as a relaxation time. Since the heat equation of this theory is of the wave-type, it automatically ensures finite speeds of propagation for heat and elastic waves. The remaining governing equations for this theory, namely, the equations of motions and constitutive relations remain the same as those for the coupled and the uncoupled theories. This theory was extended by Dhaliwal and Sherief [18] to general anisotropic media in the presence of heat sources. Recently, Sherief and Ezzat [19] have obtained the fundamental solution for this theory valid for all times.

The second generalization to the coupled theory of the thermoelasticity is what known as the theory of thermoelasticity with two relaxation times or the theory of temperature-rate-dependent thermoelasticity. Müller [20] in a review of thermodynamic of thermoelastic solids has proposed an entropy production inequality, with the help of which, he considered restrictions on a class of constitutive equations.

A generalization of this inequality was proposed by Green and Laws [21]. Green and Lindsay have obtained an explicit version of the constitutive equations in [22]. These equations were also obtained independently by Şuhubi [23]. This theory contains two constants that act as relaxation times and modifies all the equations of the coupled theory not the heat equation only. The classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry. Erbay and Şuhubi [24] studied wave propagation in finite cylinders. Ignaczak [25] studied a strong discontinuity wave and obtained a decomposition theorem for this theory [26]. Ezzat [27] has also obtained the fundamental solution for cylindrical regions. Ezzat and Othman [28] have established the model of two-dimensional equations of generalized magneto-thermoelasticity with two relaxation times in a perfectly conducting medium.

In the present paper we shall formulate the normal mode analysis developed in [28,29] to two-dimensional of thermo-viscoelasticity with two relaxation times. The resulting formulation is applied to three concrete problems. The exact expressions for temperature distribution, thermal stresses, and displacement components are obtained for each problem.

2. Formulation of the problem

We assume that there are no external forces or heat sources acting on a viscoelastic solid region. The solid is assumed to obey the equations of generalized thermo-viscoelasticity with two relaxation times, which consists of:

The equation of motion

$$\sigma_{ij,j} = \rho \ddot{u}_i. \tag{1}$$

The generalized heat conduction equation

$$kT_{,ii} = \rho C_E \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) T + \gamma T_0 \dot{e}. \tag{2}$$

The constitutive equation [15,30]

$$S_{ij} = \int_0^t R(t - \tau_0) \frac{\partial e_{ij}(\bar{x}, \tau)}{\partial \tau} \, d\tau = \hat{R}(e_{ij}) \tag{3}$$

with the assumptions

$$\sigma_{ij}(\bar{x},t) = \frac{\partial \sigma_{ij}(\bar{x},t)}{\partial t} = 0, \quad \varepsilon_{ij}(\bar{x},t) = \frac{\partial \varepsilon_{ij}(\bar{x},t)}{\partial t} = 0, \qquad -\infty < t < 0, \tag{4}$$

where

$$S_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}, \quad e_{ij} = \varepsilon_{ij} - \frac{e}{3}\delta_{ij}, \quad e = \varepsilon_{kk}, \quad \sigma_{ij} = \sigma_{ji}, \quad \bar{x} \equiv (x,y,z),$$

and $R(t)$ is the relaxation function which can be taken [16,31] in the form

$$R(t) = 2\mu\left[1 - A\int_0^t \mathrm{e}^{-\beta t} t^{\alpha^*-1}\,\mathrm{d}t\right], \tag{5}$$

where $0 < \alpha^* < 1,\ A > 0,\ \beta > 0$.

Assuming that the relaxation effects of the volume properties of the material are ignored, one can write for the generalized theory of thermo-viscoelasticity with two relaxation times

$$\sigma = K[e - 3\alpha_T(T - T_0 + \nu\dot{T})], \tag{6}$$

where $\sigma = \sigma_{ii}/3$.

Substituting Eq. (6) into Eq. (3), we obtain

$$\sigma_{ij} = \hat{R}\left(\varepsilon_{ij} - \frac{e}{3}\delta_{ij}\right) + Ke\delta_{ij} - \gamma(T - T_0 + \nu\dot{T})\delta_{ij}. \tag{7}$$

From Eqs. (1) and (7), it follows that

$$\rho\ddot{u}_i = \hat{R}\left(\frac{1}{2}\nabla^2 u_i + \frac{1}{6}e_{,i}\right) + Ke_{,i} - \gamma(T - T_0 + \nu\dot{T})_{,i}. \tag{8}$$

We shall consider only the simplest case of the two-dimensional problem. We assume that all causes producing the wave propagation are independent of the variable z and that waves are propagated only in the xy-plane. Thus all quantities were appearing in Eqs. (1)–(8) are independent of the variable z. Then the displacement vector has components $(u(x,y,t), v(x,y,t), 0)$ (plane strain problem).

Let us introduce the following non-dimensional variables:

$$x' = c_0\eta_0 x, \quad y' = c_0\eta_0 y, \quad u' = c_0\eta_0 u, \quad v' = c_0\eta_0 v, \quad t' = c_0^2\eta_o t, \quad \tau_0' = c_0^2\eta_0\tau_o,$$
$$\nu' = c_0^2\eta_0\nu, \quad \theta = \frac{\gamma(T - T_0)}{\rho c_0^2}, \quad R' = \frac{2}{3K}R, \quad \sigma_{ij}' = \frac{\sigma_{ij}}{K}.$$

In terms of these non-dimensional variables, Eqs. (2), (7) and (8), taking the following form (dropping the dashes for convenience):

$$\frac{\partial^2 u}{\partial t^2} = \hat{R}(\phi) + \frac{\partial e}{\partial x} - \left(\frac{\partial \theta}{\partial x} + v\frac{\partial^2 \theta}{\partial x \partial t}\right), \tag{9}$$

$$\frac{\partial^2 v}{\partial t^2} = \hat{R}(\psi) + \frac{\partial e}{\partial y} - \left(\frac{\partial \theta}{\partial y} + v\frac{\partial^2 \theta}{\partial y \partial t}\right), \tag{10}$$

$$\nabla^2\theta = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\theta + \varepsilon\delta_o\frac{\partial e}{\partial t}, \tag{11}$$

$$\sigma_{xx} = \hat{R}\left(\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y}\right) + e - \left(\theta + v\frac{\partial \theta}{\partial t}\right), \tag{12}$$

$$\sigma_{yy} = \hat{R}\left(\frac{\partial v}{\partial y} - \frac{1}{2}\frac{\partial u}{\partial x}\right) + e - \left(\theta + v\frac{\partial \theta}{\partial t}\right), \tag{13}$$

$$\sigma_{zz} = -\frac{1}{2}\hat{R}(e) + e - \left(\theta + v\frac{\partial \theta}{\partial t}\right), \tag{14}$$

$$\sigma_{xy} = \frac{3}{4}\hat{R}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{15}$$

where

$$\phi = \frac{\partial^2 u}{\partial x^2} + \frac{3}{4}\frac{\partial^2 u}{\partial y^2} + \frac{1}{4}\frac{\partial^2 v}{\partial x \partial y}, \tag{16}$$

$$\psi = \frac{\partial^2 v}{\partial y^2} + \frac{3}{4}\frac{\partial^2 v}{\partial x^2} + \frac{1}{4}\frac{\partial^2 u}{\partial x \partial y}, \tag{17}$$

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \tag{18}$$

3. Normal mode analysis

Eqs. (9)–(18) are simplified by decomposing the solution in terms of modes so that

$$[u, v, \theta, \phi, \psi, e, \varepsilon_{ij}, \sigma_{ij}](x, y, t) = [u^*, v^*, \theta^*, \phi^*, \psi^*, e^*, \varepsilon^*_{ij}, \sigma^*_{ij}](y)\exp(\omega t + \mathrm{i}ax). \tag{19}$$

It can be proved that

$$\hat{R}(f(x, y, t)) = \int_0^t R(t-\tau)\frac{\partial f(x, y, t)}{\partial \tau}\,\mathrm{d}\tau = \omega\bar{R}(\omega)f^*(y)\exp(\omega t + \mathrm{i}ax) \tag{20}$$

for any function $f(x, y, t)$ of class $C^{(1)}$, which satisfies the conditions

$$f(x, y, t) = \frac{\partial f(x, y, t)}{\partial t} = 0 \quad (-\infty < t < 0),$$

where

$$\bar{R}(\omega) = \int_0^\infty \mathrm{e}^{-\omega t} R(t)\,\mathrm{d}t, \tag{21}$$

and ω is the (complex) time constant and "a" is the wave number in the x-direction. This makes it possible to get

$$(D^2 - a^2)\Phi^*(y) + \omega\bar{R}(D^2 - a^2)e^*(y) = \omega^2 e^*(y), \tag{22}$$

where

$$\Phi^* = e^* - \omega_2\theta^*, \tag{23}$$

$$\alpha = \frac{1}{\omega\bar{R} + 1}. \tag{24}$$

Eq. (11) simplifies to

$$\Phi^*(y) = \frac{1}{\varepsilon_1\omega}[D^2 - a^2 - \omega_1 - \varepsilon_1\omega\,\omega_2]\theta^*(y), \tag{25}$$

where $D = \mathrm{d}/\mathrm{d}y$.

Eliminating $\Phi^*(y)$ between Eqs. (22) and (25) and using (23), we get

$$(D^4 - a_1 D^2 + a_2)\,\theta^*(y) = 0, \tag{26}$$

where

$$a_1 = \omega_1 + 2a^2 + \alpha\omega^2 + \alpha\varepsilon_1\omega\omega_2, \tag{27}$$

$$a_2 = (a^2 + \alpha\omega^2)(a^2 + \omega_1) + \alpha\omega\varepsilon_1\omega_2 a^2. \tag{28}$$

Eq. (26) can be factorized as

$$(D^2 - k_1^2)(D^2 - k_2^2)\theta^*(y) = 0, \tag{29}$$

where

$$k_{1,2}^2 = (a^2 + \omega_3) \pm \omega_4, \tag{30}$$

$$\omega_1 = \omega(1+\tau_0\omega), \quad \omega_2 = (1+v\omega), \quad \omega_3 = \frac{1}{2}[\omega_1 + \alpha\omega^2 + \alpha\varepsilon_1\omega\omega_2], \quad \omega_4 = \sqrt{a_1^2 - 4a_2}. \tag{31}$$

The solution of Eq. (29) is taken as

$$\theta^*(y) = A_1\cosh(k_1y) + A_2\cosh(k_2y) + A_3\sinh(k_1y) + A_4\sinh(k_2y), \tag{32}$$

where A_1, A_2, A_3 and A_4 are some parameters depending on "a" and "ω".

Substituting Eq. (32) into Eq. (25), we obtain

$$\begin{aligned}\Phi^*(y) = {} & \left[\frac{k_1^2 - a^2 - \omega_1 - \varepsilon_1\omega\omega_2}{\varepsilon_1\omega}\right][A_1\cosh(k_1y) + A_3\sinh(k_1y)] \\ & + \left[\frac{k_2^2 - a^2 - \omega_1 - \varepsilon_1\omega\omega_2}{\varepsilon_1\omega}\right][A_2\cosh(k_2y) + A_4\sinh(k_2y)].\end{aligned} \tag{33}$$

Substituting Eqs. (32) and (33) into Eq. (23), one obtains

$$\begin{aligned}e^*(y) = {} & \left(\frac{k_1^2 - a^2 - \omega_1}{\varepsilon_1\omega}\right)[A_1\cosh(k_1y) + A_3\sinh(k_1y)] \\ & + \left(\frac{k_2^2 - a^2 - \omega_1}{\varepsilon_1\omega}\right)[A_2\cosh(k_2y) + A_4\sinh(k_2y)].\end{aligned} \tag{34}$$

Introducing the function

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

we obtain from Eqs. (9) and (10) after some manipulations:

$$(D^2 - a^2 - \alpha_0\omega^2)\Omega^* = 0, \tag{35}$$

then

$$\Omega^*(y) = B_1\sinh(my) + B_2\cosh(my), \tag{36}$$

where

$$m^2 = a^2 + \alpha_0\omega^2, \quad \alpha_0 = \frac{4\omega}{3\bar{R}}. \tag{37}$$

Since,

$$\Omega^* = \mathrm{i}av^* - Du^*, \quad e^* = \mathrm{i}au^* + Dv^*. \tag{38}$$

From Eqs. (34), (36) and (38), we obtain

$$u^*(y) = \frac{\mathrm{i}a}{\omega\varepsilon_1}\left\{\left(\frac{k_1^2 - a^2 - \omega_1}{(k_1^2 - a^2)}\right)[A_1\cosh(k_1 y) + A_3\sinh(k_1 y)] + \left(\frac{k_2^2 - a^2 - \omega_1}{(k_2^2 - a^2)}\right)\right.$$
$$\left.\times [A_2\cosh(k_2 y) + A_4\sinh(k_2 y)]\right\} - \frac{m}{\alpha_0\omega^2}[B_1\cosh(my) + B_2\sinh(my)] \quad (39)$$

and

$$v^*(y) = \frac{1}{\omega\varepsilon_1}\left\{\frac{k_1(k_1^2 - a^2 - \omega_1)}{(k_1^2 - a^2)}[A_1\sinh(k_1 y) + A_3\cosh(k_1 y)] + \frac{k_2(k_2^2 - a^2 - \omega_1)}{(k_2^2 - a^2)}\right.$$
$$\left.\times [A_2\sinh(k_2 y) + A_4\cosh(k_2 y)]\right\} + \frac{\mathrm{i}a}{\alpha_0\omega^2}[B_1\sinh(my) + B_2\cosh(my)], \quad (40)$$

where B_1 and B_2 are some parameters depending on "a" and "ω".

Eqs. (12)–(15), in the normal mode form, are

$$\sigma_{xx}^* = \omega\bar{R}\left(\mathrm{i}au^* - \frac{1}{2}Dv^*\right) + e^* - \omega_2\theta^*, \quad (41)$$

$$\sigma_{yy}^* = \omega\bar{R}\left(Dv^* - \frac{1}{2}\mathrm{i}au^*\right) + e^* - \omega_2\theta^*, \quad (42)$$

$$\sigma_{xy}^* = \frac{3}{4}\omega\bar{R}(Du^* + \mathrm{i}av^*), \quad (43)$$

$$\sigma_{zz}^* = \left(1 - \frac{1}{2}\omega\bar{R}\right)e^* - \omega_2\theta^*, \quad (44)$$

Substituting Eqs. (32), (34), (39) and (40) into Eqs. (41)–(44), we get

$$\sigma_{xx}^*(y) = \frac{1}{\alpha\alpha_0\omega\varepsilon_1}\{\beta_1[A_1\cosh(k_1 y) + A_3\sinh(k_1 y)] + \beta_2[A_2\cosh(k_2 y) + A_4\sinh(k_2 y)]\}$$
$$- \frac{2\mathrm{i}am}{\alpha_0^2\omega^2}[B_1\cosh(my) + B_2\sinh(my)], \quad (45)$$

$$\sigma_{yy}^*(y) = \frac{1}{\alpha\alpha_0\omega\varepsilon_1}\{\beta_3[A_1\cosh(k_1 y) + A_3\sinh(k_1 y)] + \beta_4[A_2\cosh(k_2 y) + A_4\sinh(k_2 y)]\}$$
$$- \frac{2\mathrm{i}am}{\alpha_0^2\omega^2}[B_1\cosh(my) + B_2\sinh(my)], \quad (46)$$

$$\sigma_{xy}^*(y) = \frac{2\mathrm{i}a}{\alpha_o\omega\varepsilon_1}\{b_1[A_1\sinh(k_1 y) + A_3\cosh(k_1 y)] + b_2[A_2\sinh(k_2 y) + A_4\cosh(k_2 y)]\}$$
$$- \frac{(m^2 + a^2)}{\alpha_0^2\omega^2}[B_1\sinh(my) + B_2\cosh(my)], \quad (47)$$

$$\sigma^*_{zz}(y) = \frac{1}{\alpha\alpha_0\omega^2}\{b_3[A_1\cosh(k_1y) + A_3\sinh(k_1y)] + b_4[A_2\cosh(k_2y) + A_4\sinh(k_2y)]\}, \tag{48}$$

where

$$\beta_1 = \frac{(k_1^2 - a^2 - \omega_1)}{(k_1^2 - a^2)}[\alpha_0(k_1^2 - a^2) - 2\alpha k_1^2] - \alpha\alpha_0\omega_2\omega\varepsilon_1, \tag{49}$$

$$\beta_2 = \frac{(k_2^2 - a^2 - \omega_1)}{(k_2^2 - a^2)}[\alpha_0(k_2^2 - a^2) - 2\alpha k_2^2] - \alpha\alpha_0\omega_2\omega\varepsilon_1, \tag{50}$$

$$\beta_3 = \frac{(k_1^2 - a^2 - \omega_1)}{(k_1^2 - a^2)}[\alpha_0(k_1^2 - a^2) + 2\alpha a^2] - \alpha\alpha_0\omega_2\omega\varepsilon_1, \tag{51}$$

$$\beta_4 = \frac{(k_2^2 - a^2 - \omega_1)}{(k_2^2 - a^2)}[\alpha_0(k_2^2 - a^2) + 2\alpha a^2] - \alpha\alpha_0\omega_2\omega\varepsilon_1, \tag{52}$$

$$b_1 = k_1\frac{(k_1^2 - a^2 - \omega_1)}{(k_1^2 - a^2)}, \tag{53}$$

$$b_2 = k_2\frac{(k_2^2 - a^2 - \omega_1)}{(k_2^2 - a^2)}, \tag{54}$$

$$b_3 = (\alpha_0 - 2\alpha)(k_1^2 - a^2 - \omega_1) - \alpha\alpha_0\omega_2\omega\varepsilon_1, \tag{55}$$

$$b_4 = (\alpha_0 - 2\alpha)(k_2^2 - a^2 - \omega_1) - \alpha\alpha_0\omega_2\omega\varepsilon_1. \tag{56}$$

The normal mode analysis is, in fact, to look for the solution in Fourier transformed domain. Assuming that all the relations (temperature, etc.) are sufficiently smooth on the real line such that the normal mode analysis of these functions exist.

4. Applications

Problem I. A plate subjected to time-dependent heat sources on both sides [28].

We shall consider a homogeneous isotropic thermo-viscoelastic infinite thick flat plate of a finite thickness $2L$ occupying the region G given by $G = \{(x,y) \mid -\infty < x < \infty, -L \leqslant y \leqslant L, -\infty < z < \infty\}$ with the middle surface of the plate coinciding with the plane $y = 0$.

The boundary conditions of the problem are taken as:

(i) The normal and tangential stress components are zero on both surfaces of the plate; thus,

$$\sigma_{xy} = 0 \quad \text{on } y = \pm L, \tag{57}$$

$$\sigma_{yy} = 0 \quad \text{on } y = \pm L. \tag{58}$$

(ii) The thermal boundary condition

$$q_n + h_0\theta = r(x,t) \quad \text{on } y = \pm L, \tag{59}$$

where q_n denotes the normal component of the heat flux vector, h_0 is Biot's number and $r(x,t)$ represents the intensity of the applied heat sources.

Due to symmetry with respect to y-axis we can put $A_3 = A_4 = 0$ and $B_2 = 0$ in Eqs. (32)–(48). Eq. (47) together with Eq. (57) gives

$$\frac{2ia}{\alpha_0\omega\varepsilon_1}[A_1 b_1 \sinh(k_1L) + A_2 b_2 \sinh(k_2L)] - \frac{(m^2+a^2)}{\alpha_o^2\omega^2}B_1\sinh(mL) = 0. \tag{60}$$

Eq. (46) together with Eq. (58) gives

$$\frac{1}{\alpha\alpha_0\omega\varepsilon_1}\{[A_1\cosh(k_1L) + A_2\cosh(k_1L)]\} - \frac{2ia}{\alpha_0^2\omega^2}B_1\cosh(mL) = 0. \tag{61}$$

We now make use of the generalized Fourier's law of heat conduction in the non-dimensional form, [17] namely,

$$q_n + \tau_0\frac{\partial q_n}{\partial t} = -\frac{\partial\theta}{\partial n}, \tag{62}$$

by using the normal mode we get

$$q_n^* = -\frac{1}{1+\tau_0\omega}\frac{\partial\theta^*}{\partial n}. \tag{63}$$

Using Eqs. (59) and (63), we arrive at

$$\omega_1 r^*(a,\omega) = \omega_1 h_0\theta^*(y) - \omega D\theta^*(y) \quad \text{on } y = \pm L. \tag{64}$$

Using Eqs. (32) and (64), one obtains

$$A_1[\omega_1 h_0\cosh(k_1L) - \omega k_1\sinh(k_1L)] + A_2[\omega_1 h_0\cosh(k_2L) - \omega k_2\sinh(k_2L)] = \omega_1 r^*(a,\omega). \tag{65}$$

Eqs. (60), (61) and (65) can be solved for the three unknowns A_1, A_2 and B_1,

$$A_1 = \frac{\omega_1 a_{11} r^*(a,\omega)}{\Delta}, \tag{66}$$

$$A_2 = -\frac{\omega_1 a_{12} r^*(a,\omega)}{\Delta}, \tag{67}$$

$$B_1 = -\frac{2i\alpha_0\omega\omega_1 a_{15} r^*(a,\omega)}{\varepsilon_1\Delta}, \tag{68}$$

where

$$a_{11} = (m^2 + a^2)\sinh(mL)\cosh(k_2L) + 4a\alpha b_2\cosh(mL)\sinh(k_2L), \tag{69}$$

$$a_{12} = (m^2 + a^2)\sinh(mL)\cosh(k_1L) + 4a\alpha b_1\cosh(mL)\sinh(k_1L), \tag{70}$$

$$a_{13} = \omega_1 h_0\cosh(k_1L) - \omega k_1\sinh(k_1L), \tag{71}$$

$$a_{14} = \omega_1 h_0\cosh(k_2L) - \omega k_2\sinh(k_2L), \tag{72}$$

$$a_{15} = b_1\sinh(k_1L)\cosh(k_2L) - b_2\cosh(k_1L)\sinh(k_2L), \tag{73}$$

$$\Delta = a_{11}a_{13} - a_{12}a_{14}. \tag{74}$$

Problem II. A time-dependent heat punch across the surface of semi-infinite thermo-viscoelastic half-space [29].

We will consider a homogeneous isotropic thermo-viscoelastic solid occupying the region $G = \{(x,y,z)\,|\,y \leqslant 0,\ -\infty < x < \infty,\ -\infty < z < \infty\}$. In the physical problem, we shall suppress the positive exponential, which are unbounded at infinitely. Thus we should replace each $\sinh(ky)$ by $[-\frac{1}{2}\exp(ky)]$ and each $\cosh(ky)$ by $[\frac{1}{2}\exp(ky)]$.

Then, Eqs. (32), (33), (39), (40) and (45)–(48) can be written as

$$\theta^*(y) = A_1^*\exp(k_1y) + A_2^*\exp(k_2y), \tag{75}$$

$$\Phi^*(y) = \frac{1}{\varepsilon_1\omega}[\alpha_1A_1^*\exp(k_1y) + \alpha_2A_2^*\exp(k_2y)], \tag{76}$$

where

$$\alpha_1 = k_1^2 - a^2 - \omega_1 - \varepsilon_1\omega\omega_2 \quad \text{and} \quad \alpha_2 = k_2^2 - a^2 - \omega_1 - \varepsilon_1\omega\omega_2, \tag{77}$$

$$u^*(y) = \frac{\mathrm{i}a}{\omega\varepsilon_1}\left\{\frac{(k_1^2 - a^2 - \omega_1)}{(k_1^2 - a^2)}A_1^*\exp(k_1y) + \frac{(k_2^2 - a^2 - \omega_1)}{(k_2^2 - a^2)}A_2^*\exp(k_2y)\right\} - \frac{m}{\alpha_0\omega^2}B_1^*\exp(my), \tag{78}$$

$$v^*(y) = \frac{-1}{\omega\varepsilon_1}\left\{\frac{k_1(k_1^2 - a^2 - \omega_1)}{(k_1^2 - a^2)}A_1^*\exp(k_1y) + \frac{k_2(k_2^2 - a^2 - \omega_1)}{(k_2^2 - a^2)}A_2^*\exp(k_2y)\right\} - \frac{\mathrm{i}a}{\alpha_0\omega^2}B_1^*\exp(my), \tag{79}$$

$$\sigma_{xx}^*(y) = \frac{1}{\alpha\alpha_0\omega\varepsilon_1}[\beta_1A_1^*\exp(k_1y) + \beta_2A_2^*\exp(k_2y)] - \frac{2\mathrm{i}am}{\alpha_0^2\omega^2}B_1^*\exp(my), \tag{80}$$

where β_1 and β_2 are given by Eqs. (49) and (50),

$$\sigma^*_{yy}(y) = \frac{1}{\alpha\alpha_0\omega\varepsilon_1}[\beta_3 A^*_1 \exp(k_1 y) + \beta_4 A^*_2 \exp(k_2 y)] - \frac{2iam}{\alpha_0^2\omega^2} B^*_1 \exp(my), \tag{81}$$

where β_3 and β_4 are given by Eqs. (51) and (52),

$$\sigma^*_{xy}(y) = \frac{-2ia}{\alpha_o\omega\varepsilon_1}[b_1 A^*_1 \exp(k_1 y) + b_2 A^*_2 \exp(k_2 y)] + \frac{(m^2 + a^2)}{\alpha_0^2\omega^2} B^*_1 \exp(my), \tag{82}$$

$$\sigma^*_{zz}(y) = \frac{1}{\alpha\alpha_0\omega^2}[b_3 A^*_1 \exp(k_1 y) + b_4 A^*_2 \exp(k_2 y)], \tag{83}$$

where $b_1 - b_4$ are given by Eqs. (53)–(56).

The boundary conditions on the surface $y = 0$ are taken to be

$$\theta(x, 0, t) = n(x, t), \tag{84}$$

$$\sigma_{xy}(x, 0, t) = 0, \tag{85}$$

$$\sigma_{yy}(x, 0, t) = p(x, t), \tag{86}$$

where n and p are given function of x and t.

Eq. (75) together with Eq. (84) gives

$$A^*_1 + A^*_2 = n^*(a, \omega). \tag{87}$$

Eq. (82) together with Eq. (85) gives

$$\frac{2ia}{\alpha\varepsilon_1}[b_1 A^*_1 + b_2 A^*_2] - \frac{(m^2 + a^2)}{\alpha_0\omega} B^*_1 = 0. \tag{88}$$

Eq. (81) together with Eq. (86) gives

$$\alpha_0\omega[\beta_3 A^*_1 + \beta_4 A^*_2] - 2iam\varepsilon_1\alpha B^*_1 = \alpha\alpha_0^2\omega^2\varepsilon_1 p^*(a, \omega). \tag{89}$$

Solving Eqs. (87)–(89) for the unknowns A^*_1, A^*_2 and B^*_1, one obtains

$$A^*_1 = -\frac{\gamma_1}{\Delta^*}, \tag{90}$$

$$A^*_2 = \frac{\gamma_1}{\Delta^*} + n^*, \tag{91}$$

$$B^*_1 = -\frac{2ia\alpha_0\omega\gamma_2}{\alpha\varepsilon_1\Delta^*}, \tag{92}$$

where

$$\gamma_1 = 4ma^2 b_2 n^* - (m^2 + a^2)[\alpha\varepsilon_1\alpha_0\omega p^* - \beta_4 n^*], \tag{93}$$

$$\gamma_2 = \alpha\varepsilon_1\alpha_0\omega p^*(b_2 - b_1) + k_1 n^*(b_1\beta_4 - b_2\beta_3), \tag{94}$$

$$\Delta^* = 4ma^2(b_1 - b_2) - (m^2 + a^2)(\beta_4 - \beta_3). \tag{95}$$

Problem III. A plate with thermo-isolated surfaces $y = \pm L$, subjected to time dependent compression [28].

We shall consider the plate in Problem I but with the boundary conditions:

$$\frac{\partial\theta}{\partial y} = 0 \quad \text{on } y = \pm L, \tag{96}$$

$$\sigma_{xy} = 0 \quad \text{on } y = \pm L, \tag{97}$$

$$\sigma_{yy} = -P_0(x,t) \quad \text{on } y = \pm L. \tag{98}$$

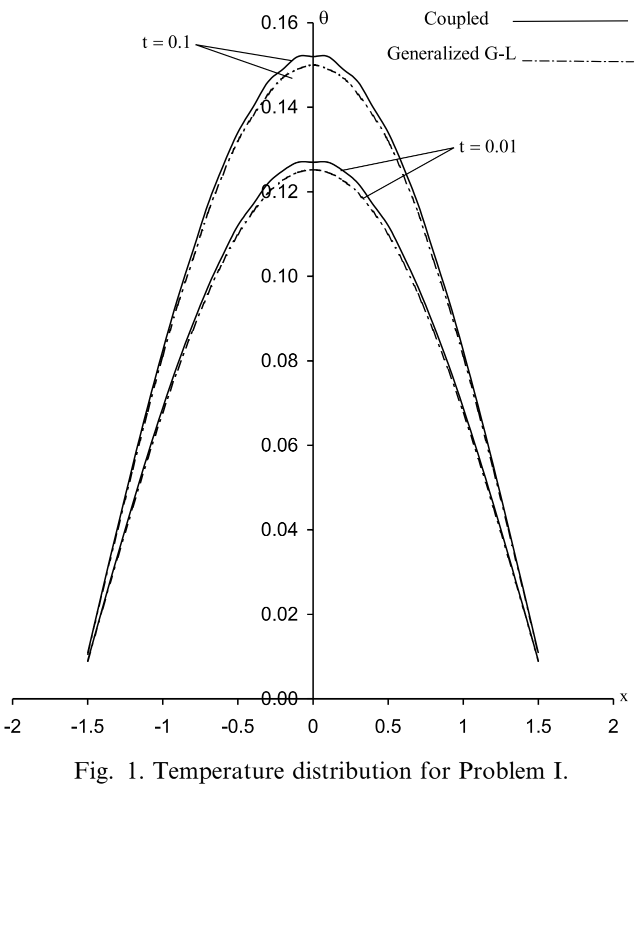

Fig. 1. Temperature distribution for Problem I.

Eq. (32) together with Eq. (96) gives

$$k_1\bar{A}_1 \sinh(k_1 L) + k_2\bar{A}_2 \sinh(k_2 L) = 0, \tag{99}$$

where $\bar{A}_1, \bar{A}_2$ are parameters depending on "a" and "ω".

Eq. (47) together with Eq. (97) gives

$$2ia\alpha_0\omega\left\{b_1\bar{A}_1 \sinh(k_1 L) + b_2\bar{A}_2 \sinh(k_2 L)\right\} - \varepsilon_1(m^2 + a^2)\bar{B}_1 \sinh(mL) = 0, \tag{100}$$

Eq. (46) together with Eq. (98) gives

$$\alpha_0\omega\left\{\beta_3\bar{A}_1 \cosh(k_1 L) + \beta_4\bar{A}_2 \cosh(k_2 L)\right\} - 2iam\alpha\varepsilon_1\bar{B}_1 \cosh(mL) = -\alpha\varepsilon_1\alpha_0^2\omega^2 P_0^*(a, \omega). \tag{101}$$

Eqs. (99)–(101) can be solved for the three unknowns $\bar{A}_1$, $\bar{A}_2$ and $\bar{B}_1$,

$$\bar{A}_1 = \frac{\alpha\alpha_0\varepsilon_1\omega(m^2 + a^2)k_2 P_0^*(a, \omega) \sinh(mL) \sinh(k_2 L)}{\bar{\Delta}}, \tag{102}$$

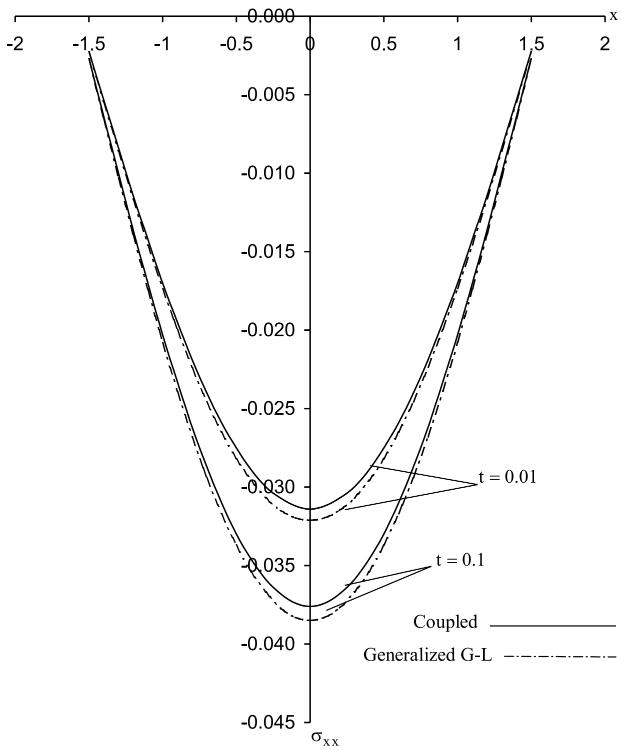

Fig. 2. Stress distribution for Problem I.

$$\bar{A}_2 = -\frac{\alpha\alpha_0\varepsilon_1\omega(m^2+a^2)k_1P_0^*(a,\omega)\sinh(mL)\sinh(k_1L)}{\bar{\Delta}}, \tag{103}$$

$$\bar{B}_1 = \frac{2\mathrm{i}a\alpha_0^2\omega^2(b_1k_2-b_2k_1)P_0^*(a,\omega)\sinh(k_1L)\sinh(k_2L)}{\bar{\Delta}}, \tag{104}$$

where

$$\bar{\Delta} = k_1c_{11}\sinh(k_1L) - k_2c_{12}\sinh(k_2L), \tag{105}$$

$$c_{11} = 4a^2m\alpha b_2\sinh(k_2L)\cosh(mL) + (m^2+a^2)\beta_4\cosh(k_2L)\sinh(mL), \tag{106}$$

$$c_{12} = 4a^2m\alpha b_1\sinh(k_1L)\cosh(mL) + (m^2+a^2)\beta_3\cosh(k_1L)\sinh(mL). \tag{107}$$

5. Numerical results

As a numerical example we have considered polymethyl methacrylate which has a wide applications in industry and medicine. Since we have $\omega = \omega_0 + \mathrm{i}\zeta$, where i is an imaginary unit,

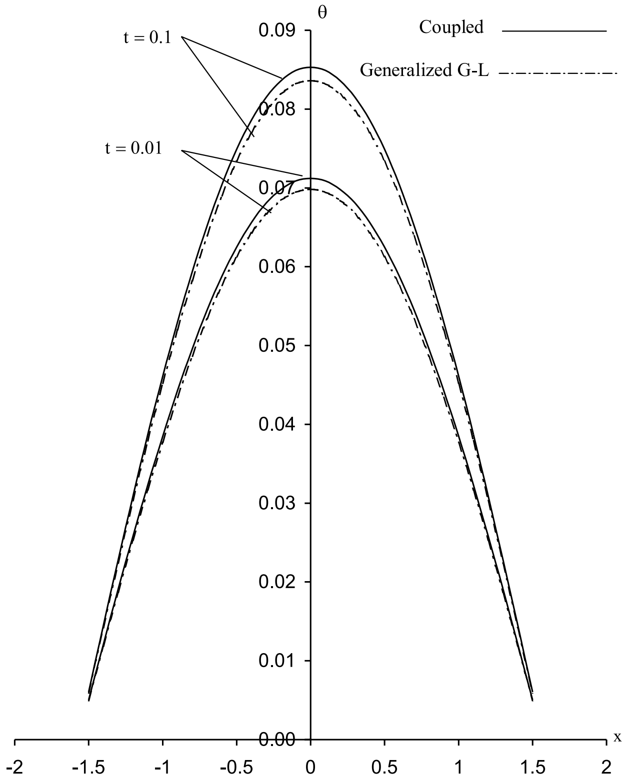

Fig. 3. Temperature distribution for Problem II.

$e^{\omega t} = e^{\omega_0 t}(\cos \zeta t + i \sin \zeta t)$ and for small values of time, we can take $\omega = \omega_0$ (real). Taking $\alpha^* = 0.5$ in Eq. (5) and using Eq. (21), we get

$$\bar{R}(\omega) = \frac{4\mu}{3K}\left[\frac{1}{\omega_0} - \frac{A\sqrt{\pi}}{\omega_0\sqrt{\omega_0 + \beta}}\right]. \tag{108}$$

The numerical constants are taken as

$$\frac{4\mu}{3K} = 0.8, \quad A = 0.106, \quad \varepsilon_1 = 0.045, \quad \beta = 0.005, \quad T_0 = 773\ \mathrm{K}, \quad \tau_0 = 0.02,$$
$$v = 0.03, \quad L = 6, \quad \omega_0 = 2, \quad \text{and} \quad \alpha = 0.59037.$$

The real part of the function $\theta(x, y, t)$ and stress component σ_{xx}, on the plane $(y = 3)$ for Problems I and III while for Problem II on $(y = -1)$, are evaluated for the two different values of time namely $t = 0.01$ and $t = 0.1$.

These results are shown in Figs. 1–6. The graph shows the four curves predicted by the different theories of thermoelasticity. In these figures the solid lines represent the solution for Lord–Shulman theory and the dotted lines represent the solution corresponding to using the coupled equation of heat conduction $(\tau_0 = v = 0)$.

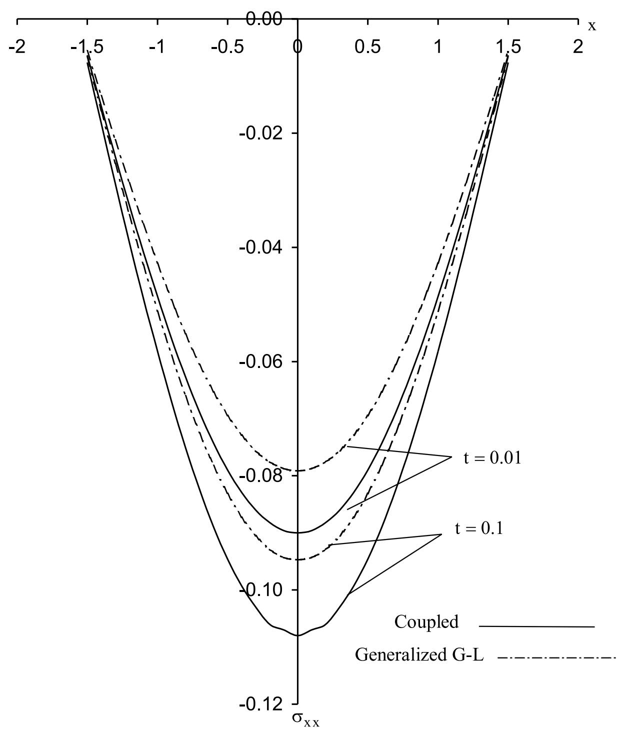

Fig. 4. Stress distribution for Problem II.

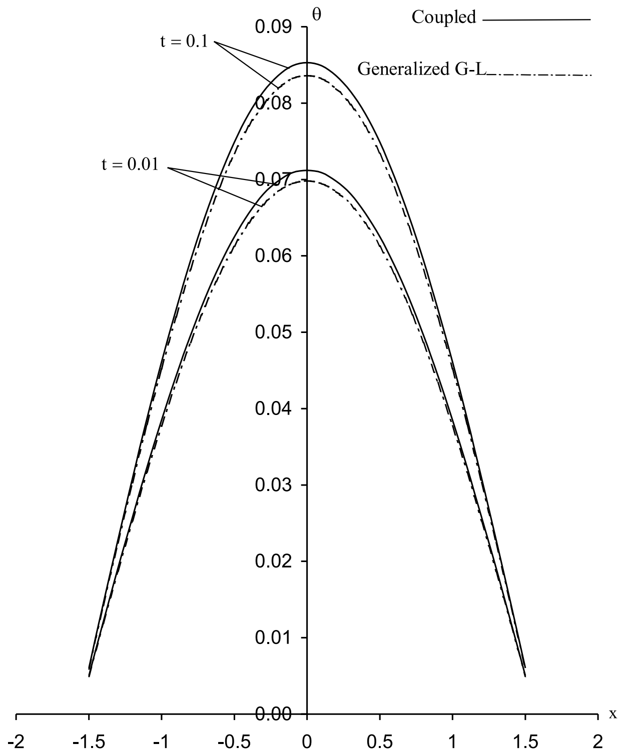

Fig. 5. Temperature distribution for Problem III.

It was found that near the surface of the solid where the boundary conditions dominate the coupled and the generalized theories give very close results. We notice also that results for the temperature and stress distributions when the relaxation time is appeared in the heat equation are distinctly different from those when the relaxation time is not mentioned in the heat equation. This is due to the fact that thermal waves in the Fourier theory of heat equation travel with an infinite speed of propagation as opposed to finite speed in the non-Fourier case. It is clear that for small values of time the solution is localized in a finite region. This region grows with increasing time and its edge is the location of the wave front. This region is determined only by the values of time t and the relaxation time τ_0 and v.

6. Concluding remarks

Owing to the complicated nature of the equations for the generalized thermo-viscoelasticity, few attempts have been made to solve problems in this field, these attempts utilize approximate methods valid for only a specific range of some parameters [15].

The state space approach developed in [32] was adopted for the solution of one-dimensional problems in generalized thermo-viscoelasticity with one relaxation time [33] and with two relaxation times [34].

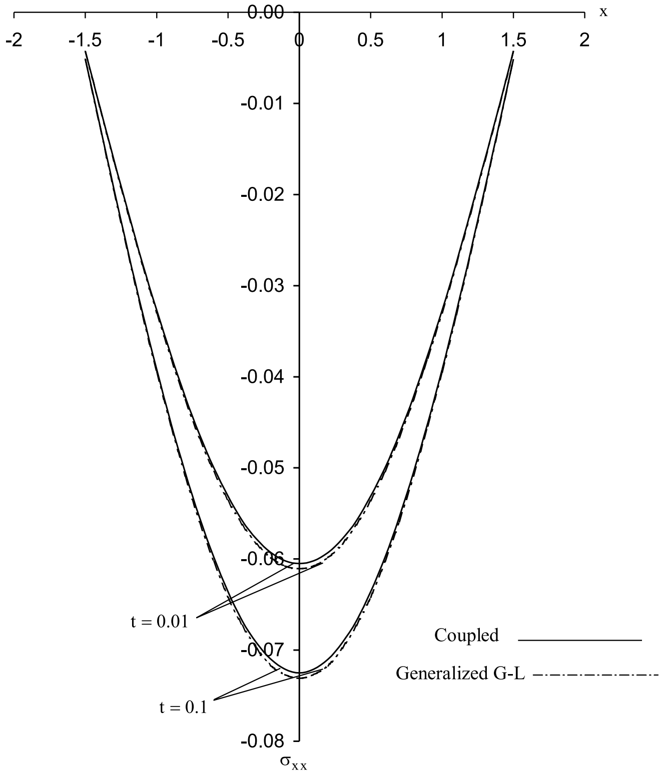

Fig. 6. Stress distribution for Problem III.

In this work, the method of normal mode analysis is introduced for the solution of two-dimensional problems in generalized thermo-viscoelasticity and applied to three specifics in which the displacement, temperature and stress are coupled. This method gives exact expressions without any assumed restrictions on either the temperature or displacement.

References

[1] B. Gross, Mathematical Structure of the Theories of Viscoelasticity, Hemann, Paris, 1953.

[2] A.J. Staverman, F. Schwarzl, in: H.A. Stuart (Ed.), Die Physik der Hochpolymeren, vol. 4, Springer, Berlin, 1956, Chapter 1.

[3] T. Alfrey, E.F. Gurnee, in: F.R. Eirich (Ed.), Rheology Theory and Applications, vol. 1, Academic Press, New York, 1956.

[4] J.D. Ferry, Viscoelastic Properties of Polymers, Wiley, New York, 1970.

[5] M.A. Biot, Theory of stress–strain relations in an isotropic viscoelasticity, and relaxation phenomena, J. Appl. Phys. 25 (1954).

[6] M.A. Biot, Variational principal in irreversible thermodynamics with application to viscoelasticity, Phys. Rev. 97 (1955).

[7] L.W. Morland, E.H. Lee, Trans. Soc. Rheol. 4 (1960).

[8] R.I. Tanner, Engineering Rheology, Oxford University Press, Oxford, 1988.

[9] R. Huilgol, N. Phan-Thien, Fluid Mechanics of Viscoelasticity, Elsevier, Amsterdam, 1997.

[10] D.R. Bland, The Theory of Linear Viscoelasticity, Pergamon Press, Oxford, 1960.
[11] M.E. Gurtin, E. Sternberg, On the linear theory of viscoelasticity, Arch. Rat. Mech. Anal. 11 (1962).
[12] E. Sternberg, On the analysis of thermal stresses in viscoelastic solids, Brown Univ. Dir, Appl. Math. TR 19 (1963).
[13] A.A. Ilioushin, The approximation method of calculating the constructures by linear thermal viscoelastic theory, Mekhanika Polimerov, Riga 2 (1968).
[14] A.A. Ilioushin, B.E. Pobedria, Mathematical Theory of Thermal Visco-elasticity, Nauka, Moscow, 1970.
[15] B.E. Pobedria, Coupled problems in continuum mechanics, J. Durabilility and Plasticity 1 (1984).
[16] M.A. Koltunov, Creeping and Relaxation, Moscow, 1976.
[17] H. Lord, Y. Shulman, A Generalized dynamical theory of thermo-elasticity, J. Mech. Phys. Solid 15 (1967) 299–309.
[18] R. Dhaliwal, H. Sherief, Generalized thermoelasticity for an isotropic media, Quart. Appl. Math. 33 (1980) 1–8.
[19] H. Sherief, M.A. Ezzat, Solution of the generalized problem of thermo-elasticity in the form of series of functions, J. Therm. Stresses 17 (1994) 75–95.
[20] I. Müller, Arch. Rat. Mech. Anal. 41 (1976) 319.
[21] A.E. Green, N. Laws, Arch. Rat. Mech. Anal. 45 (1972) 47.
[22] A.E. Green, K.A. Lindsay, Elasticity 2 (1972) 1.
[23] E.S. Şuhubi, Thermoelastic solids, in: A.C. Eringen (Ed.), Continuum Physics II, second ed., Academic Press, New York, 1975.
[24] S. Erbay, E.S. Şhuhubi, J. Therm. Stresses 15 (1986) 279.
[25] J. Ignaczak, J. Therm. Stresses 8 (1985) 25.
[26] J. Ignaczak, J. Therm. Stresses 1 (1985) 41.
[27] M.A. Ezzat, Fundamental solution in thermoelasticity with two relaxation times, Int. J. Eng. Sci. 33 (1995) 2011–2020.
[28] M.A. Ezzat, M.I. Othman, Electromagneto-thermoelastic plane waves with two relaxation times in a medium of perfect conductivity, Int. J. Eng. Sci. 38 (2000) 107–120.
[29] M.A. Ezzat, M.I. Othman, A.S. El Karamany, Electromagneto-thermoelastic plane waves with thermal relaxation in a medium of perfect conductivity, J. Therm. Stresses 24 (2001) 411–432.
[30] Y.C. Fung, Foundation of Solid Mechanics, Prentice-Hall, Englewood Cliffs, NJ, 1968.
[31] A.S. El Karamany, Deformation of a non-homogeneous viscoelastic hollow sphere, Eng. Trans. Warsaw 31 (1983) 267–271.
[32] M.A. Ezzat, State space approach to generalized magneto-thermoelasticity with two relaxation times in a medium of perfect conductivity, Int. J. Eng. Sci. 35 (1997) 741–752.
[33] M.A. Ezzat, A.S. El Karamany, A.A. Smaan, State space formulation to generalized thermo-viscoelasticity with thermal relaxation, J. Therm. Stresses 24 (2001) 823–847.
[34] M.A. Ezzat, M.I. Othman, A.S. El Karamany, State space approach to generalized thermo-viscoelasticity with two relaxation times, Int. J. Eng. Sci. 40 (2002) 283–302.